Research Notes
An Algebraic Theory of Information

Prof. Dr. Jürg Kohlas, Prof. Dr. Jürg Schmid

Institute for Informatics IIUF
University of Fribourg
mall Rue P.-A. Faucigny 2
CH – 1700 Fribourg (Switzerland)
E-mail: juerg.kohlas@unifr.ch
http://diuf.unifr.ch/tcs

Version: October 8, 2012
Contents

1 Introduction 5

2 Information Algebra 9
   2.1 Axioms ............................................. 9
   2.2 Ideal Completion .................................. 12

3 A Selection of Models 15
   3.1 An Algebra of Strings .............................. 15
   3.2 An Algebra of Partial Maps ......................... 15
   3.3 Subset Algebras .................................... 16
   3.4 Multivariate Algebras .............................. 17

4 Constructing Information Algebras 19
   4.1 Existential Quantifiers ............................. 19
   4.2 Information Systems ................................ 20
   4.3 Galois Connections ................................ 23
   4.4 Labeled Algebras .................................. 27
   4.5 Lattice Induced Algebras ......................... 38
   4.6 Morphisms ......................................... 40

5 Subalgebras and Homomorphisms 43
   5.1 Subalgebras ....................................... 43
   5.2 Homomorphisms .................................... 44
   5.3 Direct Products ................................... 46

6 Compact and Continuous Algebras 49
   6.1 Compact Information Algebras ...................... 49
   6.2 Representation Theorem ............................ 55
   6.3 Labeled Compact Algebras ......................... 60
   6.4 Continuous Information Algebras .................. 70
   6.5 Labeled Continuous Algebras ...................... 76
   6.6 Algebras Induced by Algebraic Lattices .......... 81
   6.7 Continuous Maps .................................. 84
   6.8 Cartesian Closure ................................ 88
Chapter 1

Introduction

Information is a central concept of science, especially of Computer Science. The well developed fields of statistical and algorithmic information theory focus on measuring information content of messages, statistical data or computational objects. There are however many more facets to information. We mention only three:

1. Information should always be considered relative to a given specific question. And if necessary, information must be focused onto this question. The part of information relevant to the question must be extracted.

2. Information may arise from different sources. There is therefore the need for aggregation or combination of different pieces of information. This is to ensure that the whole information available is taken into account.

3. Information can be uncertain, because its source is unreliable or because it may be distorted. Therefore it must be considered as defeasible and it must be weighed according to its reliability.

The first two aspects induce a natural algebraic structure of information. The two basic operations of this algebra are combination or aggregation of information and focusing or extraction of information. Especially the operation of focusing relates information to a structure of interdependent questions. These notes are devoted to a discussion of this algebraic structure and its variants. The algebra allows for a generic study of the structure of information.

The third item above, points to a very central property of real-life information, namely uncertainty. Clearly, probability is the appropriate tool to describe and study uncertainty. Information however is not only numerical. Therefore, the usual concept of random variables is not sufficient to study uncertain information. It turns out that the algebra of information is the
natural framework to model uncertain information. This will be another important subject of these notes.

The mathematical theory presented here is rooted in two important developments of recent years, although these developments were not originally seen as part of a theory of information. The first development is local computation, a topic starting with the quest for efficient computation methods in probabilistic networks (Lauritzen & Spiegelhalter, 1988). It has been shown around 1990 that these computation methods are closely related to some algebraic structure, for which many models or instances exist (Shafer & Shenoy, 1990; Shafer, 1991). This provides the abstract base for generic inference methods covering a wide spectrum of topics like databases, different systems of logic, probability networks and many other formalisms of uncertainty. This algebraic structure, completed by the important idempotency axiom, is the base of the mathematical theory developed here. Whereas the original structure has found some interest for computation purposes, it becomes its flavor as a theory of information only with the additional idempotency axiom (Kohlas, 2003a). This leads to information algebras, the object of the present study.

Surprisingly, the second development converging to the present work is Dempster-Shafer Theory of Evidence (Dempster, 1967a; Shafer, 1976). It was proposed originally as an extension of probability theory for inference under uncertainty, an important topic in Artificial Intelligence and related fields. In everyday language one often speaks of uncertain information. In our framework this concept is rigorously seized by random variables with values in information algebras. This captures the idea that a source emits information, which may be interpreted in different ways, which depending on various assumptions may result in different information elements. It turns out that this results in a natural generalization or extension of Dempster-Shafer Theory of Evidence, linking thus this theory in a very basic way to our theory of information (Kohlas, 2003a).

The idea that information comes in pieces, which need to be aggregated and that the information content regarding specific questions must be extracted from these pieces is elaborated and formalized in Chapter 2.1. It is argued that there is a natural order between pieces of information, reflecting information content. This is a qualitative counterpart to the numerical measures of information provided by the information theory of Shannon. In this view combination of information turns out to be represented by the order-theoretic supremum or join of two or more pieces of information. Information extraction is executed by a family of operators on pieces of informations, operators, which are related to questions, to which pieces of information pertain. The granularity of these questions induces an order between questions and their associated extraction operators, reflecting different degrees of refinement of questions. These considerations lead to an axiomatic formulation of the concept of an information algebra in Chap.
2.1. Essentially, from a mathematical abstract point of view, the algebra consists of a semilattice with a family of operators, which are *existential quantifiers*. This structure is the object of the study in these notes.

In Chap. 3 a small number of simple models or instances of information algebras are presented. This but a very restricted selection out of a plenitude of models. The practical importance of the structure of information algebras is that it provides for generic inference algorithms for seemingly very different models. For a more complete overview of instances of information algebras and generic inference algorithms we refer to (Pouly & Kohlas, 2011). There are also a number of general generic ways to construct information algebras. This is discussed in Chap. 4. These two chapters should provide a feeling for the generality of the concept of an information algebra.
Chapter 2

Information Algebra

2.1 Axioms

The algebraic model of information proposed here is based on the view that information comes in pieces and these pieces can be combined or aggregated into new pieces of information. In addition, information pertains to certain questions and provides at least partial answers to these questions. In this view the algebra to be defined in this section is based on two carrier sets, namely

1. $\Phi$, whose generic elements will be denoted by $\phi, \psi, \ldots$ and are intended to be pieces of information,
2. $D$, whose generic elements will be denoted by $x, y, \ldots$ and are thought to represent operators, which serve to extract information from the elements of $\Phi$ referring to certain questions.

Within $\Phi$ a binary operation is assumed, which models combination or aggregation of two pieces of information and which is written as $\phi \otimes \psi$. It is reasonable to assume this operation to be commutative and associative, the order in which pieces of information are combined should not matter. Also the operation must be idempotent, since the combination of a piece of information with itself generates no new information. This amounts to say that $(\Phi, \otimes)$ is a semilattice. We order $\Phi$ by $\phi \leq \psi$ if and only if $\phi \otimes \psi = \psi$, which expresses the idea that the piece of information $\phi$ contains less information than $\psi$, since it adds nothing to $\psi$. Under this order $\phi \otimes \psi$ is the supremum of $\phi$ and $\psi$ in $(\Phi, \leq)$. Consequently we call $\otimes$ join and write $\phi \otimes \psi = \phi \lor \psi$. We assume that $\Phi$ has a least element $0$ and a greatest element $1$. The element $0$ represents the vacuous information, which never adds information. The top element $1$ denotes contradiction; elements $\phi, \psi$ such that $\phi \lor \psi = 1$ are considered to be contradictory. Strictly speaking, $1$ is not a piece of information.
An operator $x \in D$ serves to extract information relative to a given question from pieces of information in $\Phi$. But the extracted information is still a piece of information, so we consider $x$ to be a map from $\Phi$ into $\Phi$ and write $\phi \mapsto x(\phi)$. Now, clearly we must have $x(\phi) \leq \phi$, since by extraction we can only loose information. Also, we assume that from the contradiction, no information can be extracted, hence $x(1) = 1$. Further from less information, less information can be extracted, such that $\phi \leq \psi$ should imply $x(\phi) \leq x(\psi)$. Less obvious, but very important, is the requirement that

$$x(x(\phi) \lor \psi) = x(\phi) \lor x(\psi).$$

It says that the extraction information according to $x$ out of the combination of a piece of information already related to $x$, $x(\phi)$, with any other piece of information, can be replaced by the combination of the pieces of information obtained after extraction. An operator with this properties is also called an existential quantifier (Cignoli, 1991) \(^1\).

Regarding any system $D$ of such operators, we assume that it is closed under composition such that we may extract information sequentially relative to different questions. The operation of composition is associative, such that $D$ forms a semigroup under composition. We require it also to be commutative, since it should not matter, in which sequence information is extracted. Finally, the semigroup must be idempotent, because extracting information relative to the same question a second time gives nothing different.

Now we may formally summarize these informal requirements of an algebraic model of information as follows:

**Definition 2.1 Information Algebra** A couple $(\Phi, D)$ satisfying the following conditions is called an information algebra:

1. $\Phi$ is a join-semilattice with 0 and 1.
2. $D$ is a commutative, idempotent semigroup of operators $\Phi \to \Phi$ under composition.
3. For all $x \in D$, the following conditions are satisfied
   
   (a) $x(1) = 1$,
   (b) $\forall \phi \in \Phi$, $x(\phi) \leq \phi$.
   (c) $\forall \phi, \psi \in \Phi$, $x(x(\phi) \lor \psi) = x(\phi) \lor x(\psi)$,

The elements of $\Phi$ are called information elements and those of $D$ domains.

---

\(^1\)In the literature the order is reversed, the semantics of information implies however the order we use here.
Again this means that $D$ is a semilattice. We define $x \leq y$ if and only if $y \circ x = x$. Then it follows that $x \circ y$ is the infimum of $x$ and $y$ in $(D, \leq)$. Therefore, we write $x \circ y = x \land y$ and $D$ becomes a meet-semilattice.

As a consequence, we have in particular that for all $\phi \in \Phi$ and for all $x, y \in D$,

$$(x \circ y)(\omega) = (x(\phi))(y(\phi)) = (x \land y)(\phi).$$

(2.1)

We remark, that this condition on the operators in $D$ conversely guarantees that $D$ is a commutative, idempotent semigroup. In examples, we shall sometimes verify this condition and prove thereby that $D$ indeed satisfies the second requirement for an information algebra.

In fact, in many specific examples of information algebras, $D$ turns out to be even a lattice (see Chapters 3 and 4).

Here follow a few immediate consequences of these axioms,

1. $x(0) = 0$,
2. $x \leq y$ implies $x(\phi) \leq y(\phi)$,
3. $\phi \leq \psi$ implies $x(\phi) \leq x(\psi)$,
4. $x(\phi) \lor \phi = \phi$,
5. $x(\phi) \lor x(\psi) \leq x(\phi \lor \psi)$.

We say that $\phi \in \Phi$ is supported by $x \in D$ or that $x$ is a support of $\phi$, if $x(\phi) = \phi$. Note that $0$ and $1$ are supported by every $x \in D$. The following properties relate to support:

1. $x(x(\phi)) = x(\phi)$,
2. $x(\phi) = \phi$ implies $x(y(\phi)) = y(x(\phi)) = y(\phi)$,
3. $x(\phi) = y(\phi) = \phi$ implies $(x \land y)(\phi) = \phi$,
4. $x(\phi) = \phi$ implies $(x \land y)(y(\phi)) = y(\phi)$,
5. $x(\phi) = \phi$ and $x \leq y$ imply $y(\phi) = \phi$,
6. $x(\phi) = \phi$ and $x(\psi) = \psi$ imply $x(\phi \lor \psi) = \phi \lor \psi$.
7. If $D$ is a lattice, then $x(\phi) = \phi$ and $y(\psi) = \psi$ imply $(x \lor y)(\phi \lor \psi) = \phi \lor \psi$.

It is left to the reader to verify these propositions.

Often, in an information algebras, each element has at least one support. Then the algebra is called supported. Note that we may always adjoin the identity mapping $id : \Phi \rightarrow \Phi$ to $D$. The system $D \cup \{id\}$ remains an
idempotent, commutative semigroup. In the domain order \( id \) becomes the top element of the augmented \( D \cup \{id\} \). Then all information elements have trivially the support \( id \) and the algebra \( (\Phi, D \cup \{id\}) \) becomes supported. In the following chapter a few instances of information algebras are presented. More follow in Chapter 4, where general, generic methods of constructing information algebras are described.

### 2.2 Ideal Completion

If a piece of information \( \phi \in \Phi \) of some information algebra \( (\Phi, D) \) is asserted, then clearly all lesser informative pieces in \( \downarrow \phi = \{\psi \in \Phi : \psi \leq \phi\} \) are automatically asserted too. More generally, if two pieces of information \( \phi, \psi \in \Phi \) are asserted, then the combined information \( \phi \lor \psi \) should be asserted too. In other words, a consistent system of pieces of information to be asserted should form an ideal.

**Definition 2.2** A non-empty set \( I \subseteq \Phi \) is called an ideal of an information algebra \( (\Phi, D) \), if

1. \( \phi \in I \) and \( \psi \in \Phi, \psi \leq \phi \), imply \( \psi \in I \),
2. \( \phi, \psi \in I \) imply \( \phi \lor \psi \).

As usual, \( \downarrow \phi \) is called a principal ideal, \( \Phi \) itself is an ideal, and if an ideal \( I \) is not equal to \( \Phi \), it is called proper. Obviously, \( 0 \) belongs to any ideal \( I \). If \( \phi \in I \), then \( x(\phi) \leq \phi \), hence \( x(\phi) \in I \) for any \( x \in D \). Now, ideals form an \( \cap \)-system, hence a complete lattice. We remark that for two ideals \( I_1 \) and \( I_2 \) of \( \Phi \),

\[
I_1 \lor I_2 = \{\psi \in \Phi : \psi \leq \phi_1 \lor \phi_2 \text{ for some } \phi_1 \in I_1, \phi_2 \in I_2\}.
\]  

(2.2)

Let \( I_{\Phi} \) be the set of ideals of \( \Phi \). We are going to extend \( I_{\Phi} \) to an information algebra by extending each operator \( x \in D \) to an operator \( \bar{x} : I_{\Phi} \to I_{\Phi} \). We define for an ideal \( I \in I_{\Phi} \),

\[
\bar{x}(I) = \{\psi \in \Phi : \psi \leq x(\phi) \text{ for some } \phi \in I\}.
\]  

(2.3)

This is still an ideal, hence \( \bar{x} \) is an operator on \( I_{\Phi} \). We claim that in this way an information algebra of ideals is created. Let \( \bar{D} \) denote the family of all operators \( \bar{x} \) for \( x \in D \).

**Theorem 2.1** If \( (\Phi, D) \) is an information algebra, then \( (I_{\Phi}, \bar{D}) \) is an information algebra and \( (\Phi, D) \) is embedded into \( (I_{\Phi}, \bar{D}) \) by the embedding \( \phi \mapsto \downarrow \phi \).
2.2. IDEAL COMPLETION

Proof. As a complete lattice, \( I_\Phi \) is a join semilattice. The improper ideal \( \Phi \) is its top element, and the ideal \( \{0\} \), its least element.

For every \( x \in D \), \( \bar{x}(\Phi) = \Phi \). Further, for all \( x \in D \) and \( I \in I_\Phi \),

\[
I \lor \bar{x}(I) = \{ \psi \in \Phi : \psi \leq \phi \lor \phi', \phi \in I, \phi' \in \bar{x}(I) \} = \{ \psi \in \Phi : \psi \leq \phi \lor x(\phi'), \phi, \phi' \in I \}
\]

This shows that \( I \lor \bar{x}(I) \subseteq I \). The converse inclusion is evident, hence \( \bar{x}(I) \subseteq I \). Thus \( \bar{x} \) satisfies condition (b). Condition (c) is verified as follows, using condition (c) in the information algebra \((\Phi, D)\):

\[
\bar{x}(\bar{x}(I_1) \lor I_2) = \{ \psi \in \Phi : \psi \leq x(\phi), \phi \in \bar{x}(I_1) \lor I_2 \} = \{ \psi \in \Phi : \psi \leq \phi_1' \lor \phi_2', \phi_1' \in \bar{x}(I_1), \phi_2 ' \in I_2 \} = \{ \psi \in \Phi : \psi \leq \phi_1' \lor \phi_2', \phi_1' \in \bar{x}(I_1), \phi_2 ' \in \bar{x}(I_2) \} = \bar{x}(I_1) \lor \bar{x}(I_2).
\]

To check that \( D \) forms a commutative, idempotent semigroup under composition, we verify (2.1), using this same condition in \((\Phi, D)\):

\[
\bar{x}(\bar{y}(I)) = \{ \psi \in \Phi : \psi \leq x(\phi'), \phi' \in y(I) \} = \{ \psi \in \Phi : \psi \leq x(y(\phi)) = (x \land y)(\phi), \phi \in I \} = (x \land y)(I).
\]

So, \((I_\Phi, D)\) is an information algebra.

The map \( \phi \mapsto \downarrow \phi \) is clearly one-to-one. Further by (2.2)

\[
\phi \lor \psi \mapsto \downarrow (\phi \lor \psi) = \downarrow \phi \lor \downarrow \psi
\]

Also, we have \( \psi \in \bar{x}(\downarrow \phi) \) if and only if \( \psi \leq x(\phi) \), which in turn holds, if and only if \( \psi \in \downarrow x(\phi) \). Therefore \( x(\phi) \mapsto \bar{x}(\downarrow \phi) \). Finally \( 0 \mapsto \{0\} \) and and \( 1 \mapsto \Phi \). So this map is an embedding (see also Section 5.2).

In abuse of notation we shall henceforth identify \( \Phi \) with its image \( I(\Phi) \) under the mapping \( I : \phi \mapsto \downarrow \phi \). We also identify \( D \) with \( \bar{D} \) and write \( x(I) \) instead of \( \bar{x}(I) \). Also \( \phi \in I \) is then often written as \( \phi \leq I \), referring to the order in \( I_\Phi \).

The information algebra \((I_\Phi, D)\) is called the ideal completion of the information algebra \((\Phi, D)\).
Chapter 3

A Selection of Models

3.1 An Algebra of Strings

Strings over a finite set Σ as alphabet provide a first very simple model of an information algebra. Consider $\Sigma^{**} = \Sigma^* \cup \Sigma^\omega \cup \{z\}$, where $\Sigma^*$ is the set of strings of finite length, including the empty string $\epsilon$, $\Sigma^\omega$ is the set of infinite strings, and $z$ is an adjoined element, $z \notin \Sigma^* \cup \Sigma^\omega$. Define an order on $\Sigma^{**}$ by $r \leq s$ if $r$ is a prefix of $s$ or $s = z$. Further denote by $s^{|n|}$ the prefix of length $n$ of the string $s$, provided $s$ has length of at least $n$. The set $\Sigma^{**}$ becomes a join-semilattice if join is defined as follows:

$$r \lor s = \begin{cases} s & \text{if } r \leq s, \\ r & \text{if } s \leq r, \\ z & \text{otherwise.} \end{cases}$$

The element $z$ is the top element in this semilattice, $r \leq z$ for any string $r$, and the empty string $\epsilon$ is the bottom element. In addition, take the lattice $D = \{0, 1, 2, \ldots\} \cup \{\infty\}$ ordered under its natural order and define for $n \in \{0, 1, 2, \ldots\}$ the operators for $n \neq z$,

$$n(s) = \begin{cases} s^{|n|} & \text{if } n \leq |s|, \\ s & \text{if } n \geq |s|. \end{cases}$$

Further, define $\infty(s) = s$ and $n(z) = z$. The system $(\Sigma^{**}, \omega)$ is an information algebra.

3.2 An Algebra of Partial Maps

Consider any set $X$. A partial map on $X$ is a map $\sigma : S \rightarrow X$, where $S$ is a subset of $X$. The set $S$ is the domain of the partial map $\sigma$ and denoted by $d(\sigma)$. The set of all partial maps on $X$ together with an adjoined element $z$ is denoted by $X \rightarrow o \rightarrow X$. Partial maps on $X$ are ordered by $\sigma \leq \tau$ if
d(σ) ⊆ d(τ) and σ(x) = τ(x) for all x ∈ d(σ), or τ = z. The set $X \rightarrow o \rightarrow X$ becomes a join semi-lattice if join is defined as follows:

$$\sigma \lor \tau = \begin{cases} 
\sigma & \text{if } \tau \leq \sigma, \\
\tau & \text{if } \sigma \leq \tau, \\
z & \text{otherwise.}
\end{cases}$$

The element z is the top element of the semilattice and the empty function $\epsilon$, defined on the empty set, is the bottom element. In addition, consider the lattice $\mathcal{P}(X)$ of all subsets of $X$. For any subset $S \in \mathcal{P}(X)$ and $\sigma \in X \rightarrow o \rightarrow X$ define

$$S(\sigma) = \sigma|d(\sigma) \cap S.$$ 

For the element z define $S(z) = z$. The system ($X \rightarrow o \rightarrow X, \mathcal{P}(X)$) is an information algebra.

### 3.3 Subset Algebras

Let $U$ be any set, called the *universum*, and let $D$ be a lattice of partitions of $U$. A partition $P$ is usually considered as smaller than another partition $Q$, $P \leq Q$, if every block of $P$ is subset of a block of $Q$, that is the partition $P$ is finer than $Q$. For our purposes, the opposite order is more appropriate, the finer partition allows more possible answers. In the following we assume this opposite order between partitions, such that the usual join of partitions becomes meet for us.

A subset $S$ of $U$ is said to be saturated by a partition $P$, if $S$ is the union of blocks of $P$. Let $\Phi_P$ denote the family of all subsets $S$ saturated by $P$ and

$$\Phi = \bigcup_{P \in D} \Phi_P.$$ 

Now, $\Phi$ is closed under finite intersection, and hence a $\cap$-semilattice. In fact, the intersection of a set $S$, saturated by $P$ and a set $T$, saturated by $Q$, is a set saturated by $P \lor Q$, in the opposite of the usual order. Again, we consider this as a *join* in the information order, since $S$ is more informative than $T$, if $S \subseteq T$, $S$ constrains the possible answers more than $T$. So we have the information order $T \leq S$ if $S \subseteq T$ and in this information order

$$S \lor T = S \cap T.$$ 

Then $\Phi$ becomes in this way a join-semilattice.

Further, for any partition $P \in D$, we define the saturation operator induced by $P$ as

$$\mathcal{P}(S) = \bigcup\{B \in P : B \cap S \neq \emptyset\}$$
for all $S \in \Phi$. So, $P(S)$ is saturated with respect to the partition $P$, hence $P(S) \in \Phi$. The system $(\Phi, D)$ as defined here, is an information algebra. In verifying this, the opposite order of partitions must be taken into account.

### 3.4 Multivariate Algebras

In many practical applications a set of variables is considered and the information pertains to the unknown values of these variables. So, let $I$ be an index set of variables $X_i, i \in I$ and let $\Omega_i$ denote the set of possible values, the domain of variable $X_i$. For a subset $s$ of $I$ define

$$\Omega_s = \prod_{i \in s} \Omega_i,$$

the domain of the variables $X_i$ for $i \in s$. Consider subsets $S$ of $\Omega_I$ with elements $x, y, \ldots$. For a subset $s$ of $I$ and $x \in \Omega_I$, let $x[s] = y \in \Omega_s$ such that $x_i = y_i$ for all $i \in s$. Define equivalence relations $\equiv_s$ in $\Omega_I$ for any $s \subseteq I$, by $x \equiv_s y$ if $x[s] = y[s]$. The set

$$s(S) = \{x \in \Omega_I : \exists y \in S, y \equiv_s x\}$$

is called the $s$-cylindrification of $S$. A set $S$ is called an $s$-cylinder if $S = s(S)$.

Consider now any lattice $D$ of subsets of $I$, let $\Phi_s$ be the family of $s$-cylinders in $\Omega_I$ and let

$$\Phi = \bigcup_{s \in D} \Phi_s.$$

It can be seen that $\Phi$ is closed under finite intersection. In fact the intersection $S \cap T$ of an $s$-cylinder and a $t$-cylinder is an $s \cup t$ cylinder. So, $\Phi$ is an intersection-semilattice. For the same reason as in the previous section, the information order is rather the opposite of set inclusion, $S \leq T$ if $T \subseteq S$. Then intersection becomes join in this opposite order, and $\Phi$ is in this sense a join-semilattice. Further the empty set is the bottom element and $\Omega_I$ the top element of the join-semilattice.

For any $S \in \Phi$ and $s \in D$, we consider the operation of $s$-cylindrification, $s(S)$. With this convention $(\Phi, D)$ becomes an information algebra. In fact, it is a special case of the algebras discussed in Section 3.3 with saturation replaced by cylindrification. In fact, any subset $s$ of $I$ defines a partition of $\Omega_I$ with the equivalence classes of the equivalence relation $\equiv_s$ as blocks.

Often all the variables have real numbers as domains, $\Omega_i = \mathbb{R}$. Then $\Omega_I = \mathbb{R}^n$, if only a finite number of variables is considered. In this case, special classes of subsets like affine subspaces, convex sets or convex polyhedra can be considered(Kohlas, 2003a). All these examples provide information algebras, in fact subalgebras of the algebra of all subsets discussed in this section.
Chapter 4

Constructing Information Algebras

4.1 Existential Quantifiers

A monadic algebra is a Boolean algebra $\Phi$ together with a map $\exists: \Phi \to \Phi$, such that

1. $\exists(1) = 1$,
2. $\forall \phi \in \Phi, \exists(\phi) \leq \phi$,
3. $\forall \phi, \psi \in \Phi, \exists(\exists(\phi) \lor \psi) = \exists(\phi) \lor \exists(\psi)$,

see (Halmos, 1962; Plotkin, 1994). A map $\exists$, which satisfies these three requirements is called an existential quantifier, which corresponds to the terminology used in Section 2.1. These requirements concern only the join-operation, therefore apply also to a join-semilattice $\Phi$, it need not be Boolean. Consider now a finite set $V$ such that for any $i \in V$ there is an existential quantifier $\exists_i$ on $\Phi$ and

$$\forall i, j \in V, \forall \phi \in \Phi, \exists_i(\exists_j(\phi)) = \exists_j(\exists_i(\phi)).$$

Then we may define a map $\exists(s)$ for all subsets of $s = \{i_1, \ldots, i_n\}$ of $V$ by

$$\exists_s(\phi) = \exists_{i_n}(\exists_{i_{n-1}}(\ldots \exists_{i_1}(\phi)\ldots))$$

Clearly $\exists(s)$ is an existential quantifier on $\Phi$ and for $s, t$ subsets of $V$,

$$\exists_{s \cup t}(\phi) = \exists_s(\exists_t(\phi)).$$

In this way, we obtain a quantifier algebra.

Consider now $(\Phi, \mathcal{P}(V))$, where $\mathcal{P}(V)$ is the lattice of all subsets of $V$, and define for all $s \in \mathcal{P}(V)$ an operator $\Phi \to \Phi$ by

$$s(\phi) = \exists_{V \setminus s}(\phi).$$
Then, with these operators, \((\Phi, \mathcal{P}(V))\) is an information algebra. In this way any quantifier algebra gives rise to an information algebra: existential quantification, in specific applications, elimination of variables, corresponds to extracting information. Examples of this type of information algebras are in this way obtained as reducts of cylindric algebras (Henkin et al., 1971).

### 4.2 Information Systems

The idea is to present pieces of information with propositions expressed by sentences of some unspecified formal language. The internal structure, the grammar of these sentences is of no concern here. The essential point here is that there is an entailment relation which permits from a set of sentences to deduce further sentences. We shall show how to construct an information algebra in this context.

Let \(L\) be a set of sentences, called the language. An entailment relation relates sets of sentences \(X \subseteq L\) to single sentences \(s \in L\):

**Definition 4.1** A relation \(X \vdash s\) between any subset \(X \subseteq L\) and elements \(s \in L\) is called an entailment relation, if the following two conditions are satisfied:

1. \((E1)\) \(X \vdash s\) for all \(s \in X\),
2. \((E2)\) If \(X \vdash s\) for all \(s \in Y\) and \(Y \vdash t\), then \(X \vdash t\).

The relation \(X \vdash s\) means that sentence \(s\) can be deduced form the set \(X\).

A pair \((L, \vdash)\) of a language together with an entailment relation is called an information system. This is similar to Scott’s information systems (Davey & Priestley, 1990), although no consistent sets of sentences are singled out and no finiteness conditions are considered so far, see however Example 6.5 in Section 6.1.

Associated with an entailment relation is an operator \(C\) mapping the power set \(\mathcal{P}(L)\) into itself. The set \(C(X)\) is the set of all sentences derivable from \(X\) by the entailment relation,

\[
C(X) = \{ s \in L : X \vdash s \}.
\]

This operator \(C\) is the consequence operator of the entailment relation, it is also a closure operator, that is an operator, satisfying the three conditions of the following lemma:

**Lemma 4.1** The operator \(C\) satisfies for all subsets \(X, Y\) of \(L\) the conditions

1. \((C1)\) \(X \subseteq C(X)\),
2. \((C2)\) \(C(C(X)) = C(X)\),
3. \((C3)\) If \(X \subseteq Y\), then \(C(X) \subseteq C(Y)\).
4.2. INFORMATION SYSTEMS

Proof. (C1) follows from (E1).

(C2) We need only to show that $C(C(X)) \subseteq C(X)$, since the converse inclusion holds by (C1). If $s \in C(C(X))$, then $C(X) \vdash s$. Since $X \vdash t$ for all $t \in C(X)$, (E2) implies $X \vdash s$, hence $s \in C(X)$.

(C3) Let $s \in C(X)$, hence $X \vdash s$. Now $Y \vdash t$ for all $t \in X$ and thus by (E2) $Y \vdash s$, therefore $s \in C(Y)$.

Sets $X$ such that $X = C(X)$ are called closed. If a set of sentences $X$ of the language $L$ is asserted, then the set of all its consequences, the closure $C(X)$ can be regarded as the information conveyed by $X$. So, we consider the closed sets as pieces of information within the information system $(L, \vdash)$.

Let $\Phi$ denote the set of all closed sets in $L$. It is a $\cap$-system, since the intersection of any family of closed set is closed. It contains a least element $C(\emptyset)$, the tautologies, and a top element, namely $L$ itself. So, $\Phi$ is a complete lattice, and in particular a join-semilattice with $0 = C(\emptyset)$ and $1 = L$, where,

$$C(X) \lor C(Y) = C(C(X) \cup C(Y)).$$

The following property of a consequence operator is often used:

**Lemma 4.2** For any closure operator, for all subsets $X, Y$ of $L$, we have

$$C(X \cup Y) = C(C(X) \cup Y).$$

**Proof.** We show the $C(C(X) \cup Y) \subseteq C(X \cup Y)$, the reverse inclusion being obvious from (C1) and (C3). From $X, Y \subseteq X \cup Y$ it follows that $C(X) \subseteq C(X \cup Y)$ and $Y \subseteq C(X \cup Y)$, hence $C(X) \cup Y \subseteq C(X \cup Y)$. Applying the consequence operator on both sides of this inclusion and using (C3) and (C2) we obtain $C(C(X) \cup Y) \subseteq C(X \cup Y)$. 

So we have also that

$$C(X) \lor C(Y) = C(X \cup Y).$$

Hence, in order to combine two pieces of information, asserted by statements $X$ and $Y$, the closure of the union of the statements $X \cup Y$ is to be taken.

In many cases one is only interested in the information relative to a selected sublanguage $M \subseteq L$. This restricted information is simply $C(X) \cap M$. This set is however no more closed in $L$, that is $C(X) \cap M \neq C(C(X) \cap M)$. The latter represents thus the information extracted from $X$ relative to the sublanguage $M$. We introduce the operator $C_M : \Phi \to \Phi$ defined by $C_M(C(X)) = C(C(X) \cap M)$. We would like this operator to be an existential operator, an operator satisfying (a) to (c) of item 3 of the axioms of an information algebra (see Section 2.1). This requires that the selection of the sublanguage $M$ by itself conveys not already information. This means that $C(M) = L$. Then $C_M(L) = C(M) = L$. So condition (a) is satisfied.
Condition (b) is also satisfied, since $C_M(C(X)) = C(C(X) \cap M) \subseteq C(X)$. Condition (c) requires that

$$C_M(C(C(X) \cup Y)) = C(C(C(X) \cup C(M(C(Y))))$$

This is equivalent to the following condition on the consequence operator:

$$(C4) \quad C((C(X) \cap M) \cup Y) \cap M = C((C(X) \cap M) \cup C(C(Y) \cap M)) \cap M.$$ 

We refer to (Kohlas, 2003a) for a proof of this equivalence.

In order to construct an information algebra, we are going to consider a family $S$ of sublanguages such that

1. $S$ is closed under finite intersection: If $M_1, M_2 \in S$ then $M_1 \cap M_2 \in S$.
2. For each $M \in S$ the operator $C_M$ satisfies (C4).

According to these requirements, $S$ is a meet-semilattice under set-inclusion.

Let $D$ be the family of all operators $C_M$ for $M \in S$. We require further in view of item 2 of the axioms of an information algebra (see Section 2.1) that the family of these operators satisfy for any two sublanguages $L, M \in S$

$$C_L \circ C_M = C_M \circ C_L = C_{L \cap M}$$

similar to (2.1). Then, the operators form a commutative, idempotent semigroup, hence a semilattice of existential quantifiers, see Section 2.1. The condition (4.1) can be expressed in a requirement for the consequence operator $C$. In fact, using the definition of $C_M$, we obtain, that for all $L, M \in S$ and for all subsets $X$ of sentences

$$(C5) \quad C(C(C(X) \cap M) \cap L) = C(C(C(X) \cap L) \cap M) = C(C(X) \cap L \cap M).$$

We note that this condition is characterised by an interpolation property.

The consequence operator $C$ has the interpolation property with respect to a semilattice of sublanguages $S$, if, for all $L, M \in S$, from

$$s \in L, \quad X \subseteq M, \quad \text{and} \quad s \in C(X)$$

it follows that there exists a set of sentences $Y \subseteq L \cap M$ such that $Y \subseteq C(X)$ and $s \in C(Y)$. The set of sentences $Y$ is called the interpoland between $X$ and $s$. Now, $C$ satisfies (4.2) if and only if it has the interpolation property relative to $S$ (Kohlas & Staerk, 2007; Kohlas, 2003a).

Finally, we see that under these conditions $(\Phi, D)$ with $D$ the commutative semigroup of operators $C_L$ for $L \in S$ is an information algebra derived from the information system $(L, \vdash)$ together with the family $S$ of sublanguages of $L$.

Conversely, it can be shown that any information algebra $(\Phi, D)$ induces an information system together with a family of sublanguages (Kohlas, 2003a). This is quite similar to the equivalence between domains and Scott information systems in domain theory (Davey & Priestley, 1990).
4.3 Galois Connections

Information about possible worlds or models is often also expressed by some language. This idea can be seized by Galois connections and it extends somewhat the concept of an information system (see Section 4.2). Consider two sets $L$ and $M$ together with a relation $|\subseteq L \times M$. The set $L$ is thought as a set of sentences or formulae, whereas the elements of set $M$ are considered as models. The expression $(s,m) \in |$ means that $s$ models $m$ or $m$ satisfies $s$, that is, $m$ is a model of $s$. Subsequently we write $m | s$ instead of $(s,m) \in |$. The triple $(L,M,|)$ determines a Galois connection. We assume only that each $s \in L$ has a model, and each model $m \in M$ satisfies a sentence. As an illustrating example we present here the case of predicate logic.

Example 4.1 Predicate Logic. We start with defining the language, the set of formulae $L$. The vocabulary of $L$ consists of a countable set of variables $X_1, X_2, \ldots$ and a countable set of predicate symbols $P_1, P_2, \ldots$. It further includes the logical constants $\bot$ and $\top$ and the connectors $\land, \lor, \exists$. Each predicate symbol $P_i$ has a definite rank $\rho_i$, and a predicate symbol $P$ with rank $\rho$ is referred to as a $\rho$-place predicate. Formulae of predicate logic are built according to the following rules:

1. $P_i X_{i_1} \ldots X_{i_{\rho_i}}$, where $\rho$ is the rank of $P_i$, $\bot$ and $\top$ are (atomic) formulae.

2. If $f$ is a formulae, then $\neg f$ and $(\exists X_i)f$ are formulae.

3. If $f$ and $g$ are formulae, then $f \land g$ is a formula.

The predicate language $L$ consists of all formulae which are obtained by applying these rules a finite number of times $^1$.

In order to define an associated model structure $M$ we define an interpretation of formulae of the predicate language $L$. For this purpose we choose a relational structure $R = \{U, R_1, R_2, \ldots\}$ where $U$ is a nonempty set, called the universe, and the $R_i$ are relations among the elements of $U$ with arity $\rho_i$ equal the rank of the predicate $P_i$. In other words, $R_i$ are subsets of the cartesian product $U^{\rho_i}$. A valuation is a mapping $v : \{1, 2, \ldots\} \rightarrow U$, which assigns to each variable $X_i$ a value $v(i) \in U$. We write $U^\omega$ for the set of all possible valuations. Given a valuation $v$ and an index $i$, we define the set of all valuations that agree $v$ on all indices, except $i$,

$$v^{\triangleright i} = \{u \in U^\omega : u(j) = v(j) \text{ for } j \neq i\}.$$  

The model set $M$ is the set $U^\omega$ of all possible valuations.

---

$^1$Note that sentences or formulae in $L$ are not necessarily closed formulae without free variables, that is, sentences in the restricted sense often used in predicate logic.
Valuations $\mathbf{v}$ are used to assign a truth value $\hat{\mathbf{v}}(f) \in \{0, 1\}$ to a formula $f \in \mathcal{L}$. The assignment is defined inductively as follows:

1. $\hat{\mathbf{v}}(\top) = 1$ and $\hat{\mathbf{v}}(\bot) = 0$;
2. $\hat{\mathbf{v}}(P, X_{i_1}, \ldots, X_{i_\rho}) = 1$ if $\mathbf{v}(i_1, \ldots, i_\rho) \in R_i$ and $\hat{\mathbf{v}}(P, X_{i_1}, \ldots, X_{i_\rho}) = 0$ otherwise;
3. $\hat{\mathbf{v}}(\neg f) = 1$ if $\hat{\mathbf{v}}(f) = 0$ and $\hat{\mathbf{v}}(\neg f) = 0$ otherwise;
4. $\hat{\mathbf{v}}((\exists X_i)f) = 1$ if there is a valuation $\mathbf{u} \in \mathbf{v} \Rightarrow i$ such that $\hat{\mathbf{u}} = 1$ and $\hat{\mathbf{v}}((\exists X_i)f) = 0$ otherwise;
5. $\hat{\mathbf{v}}(f \land g) = 1$ if $\hat{\mathbf{v}}(f) = \hat{\mathbf{v}}(g) = 1$ and $\hat{\mathbf{v}}(f \land g) = 0$ otherwise.

A valuation $\mathbf{v}$ is called a model of $f$ in the structure $\mathcal{R}$ if $\hat{\mathbf{v}}(f) = 1$. This is written as $\mathbf{v} \models f$ and that is the relation in the Galois connection $(\mathcal{L}, \mathcal{M}, \models)$ of predicate logic.

Define for a set $S \subseteq \mathcal{L}$ of sentences and for a set $M \subseteq \mathcal{M}$ of models

\[ \hat{r}(S) = \{m \in \mathcal{M} : m \models s \ \forall s \in S\}; \quad \hat{r}(M) = \{s \in \mathcal{L} : m \models s \ \forall m \in M\}. \]

The set $\hat{r}(S)$ contains all the models satisfying all sentences in $S$. In other word, $\hat{r}(S)$ is the information about models, expressed by $S$. On the other hand, $\hat{r}(M)$ contains all the sentences satisfied by all models in $M$. It is the theory of $M$. We have $\hat{r}(\emptyset) = \mathcal{L}$ and assume that no model satisfies all sentences, $\hat{r}(\mathcal{L}) = \emptyset$. It is well known that the operators $C_\models$ and $C^\models$ defined by

\[ C_\models(S) = \hat{r}(\hat{r}(S)), \quad C^\models = \hat{r}(\hat{r}(M)) \]

are consequence operators, hence closure operators, (Wojcicki, 1988). Sets $S \subseteq \mathcal{L}$ and $M \subseteq \mathcal{M}$ such that $C_\models(S) = S$ and $C^\models(M) = M$ are called $\models$-closed. We call a $\models$-closed sets, $\hat{r}(S) = C_\models(\hat{r}(S))$ an information and $\hat{r}(M) = C^\models(\hat{r}(M))$ a theory. It is well-known that the families $C(\mathcal{L})$ and $C(\mathcal{M})$ of $\models$-closed sets of information pieces in $\mathcal{L}$ or theories in $\mathcal{M}$ respectively form complete lattices under inclusion with intersection as meet, (Davey & Priestley, 1990).

Under information order however, we would consider a theory with a smaller model set as more informative. So we define the information order in $C(\mathcal{M})$ as $M \subseteq M'$ if $M \supseteq M'$. In this order intersection of closed model sets is rather considered as join. In this sense, $C(\mathcal{M})$ forms a join-semilattice as required for an information algebra with

\[ M \lor M' = M \cap M'. \]

In this order $\mathcal{M}$ is the bottom element and $\emptyset$ the top element.
The missing element for an information algebra is a semilattice of information-extraction operators. The inspiration for this comes from the subset information algebra based on a lattice of partitions of an universe. Here too we may consider partitions of the model set \( \mathcal{M} \), which describe certain coarsenings of the original models in \( \mathcal{M} \). Consider therefore a semilattice \( \mathcal{D} \) and indexed equivalence relations in \( \mathcal{M} \) with elements of \( x, y \in \mathcal{D} \) such that

\[
\equiv_{x \wedge y} = \text{sup}\{\equiv_x, \equiv_y\}, \tag{4.3}
\]

where \( \text{sup} \) denotes here the ordinary supremum in the lattice of equivalence relations on \( \mathcal{M} \). Requiring only that \( x \leq y \) implies \( \equiv_x \supseteq \equiv_y \), this condition follows from the following requirement:

\[
m \equiv x \wedge y n \text{ implies that there is a } l \text{ such that } m \equiv_x l, n \equiv_y l. \tag{4.4}
\]

In fact, by the monotonicity of the indexing of the equivalence relations, \( \equiv_{x \wedge y} \supseteq \text{sup}\{\equiv_x, \equiv_y\} \). On the other hand, from (4.4) and the monotonicity of the indexing it follows that \((m, n) \in \text{sup}\{\equiv_x, \equiv_y\}\), hence \( \equiv_{x \wedge y} \subseteq \text{sup}\{\equiv_x, \equiv_y\} \) which shows the equality (4.3).

For every \( x \in \mathcal{D} \) we define the \( x\)-cylindrification of a subset \( M \) of models as follows:

\[
x(M) = \{m \in \mathcal{M} : \exists n \in M, n \equiv_x m\}.
\]

Clearly, \( x(M) \) is the saturation of \( M \) with respect to the partition of equivalence classes of \( \equiv_x \). We require further, that for any \( \models \)-closed set \( M \in \mathcal{C}(\mathcal{M}) \) the cylindrification \( x(M) \) is also \( \models \)-closed. Then, for all \( x \in \mathcal{D} \), \( x \) are operators on the semilattice \( \mathcal{C}(\mathcal{M}) \). In order to show that \((\mathcal{C}(\mathcal{M}), D)\) is an information algebra, conditions (a) to (c) of item 3 in the axioms of an information algebra (see Section 2.1) of operators have to be verified: Clearly conditions (a) and (b) are satisfied. Condition (c) is also easily checked:

\[
x(x(M) \cap N) = \{m : m \equiv_x n, n \in x(M) \cap N\} = \{m : m \equiv_x n, n \in x(M)\} \cap \{m : m \equiv_x n, n \in N\} = x(M) \cap x(N).
\]

In order to verify the semigroup properties we use (4.4): We have \( (x \wedge y)(M) = \{m : m \equiv_{x \wedge y} n, n \in M\} \). So there is an \( l \) such that \( n \equiv_y l \), hence \( l \in y(M) \), and \( m \equiv_x l \), hence \( m \in x(y(M)) \). Conversely, \( m \in x(y(M)) \), implies \( m \equiv x l \equiv y n, n \in M \), hence also \( m \equiv_{x \wedge y} l \equiv_{x \wedge y} n, n \in M \), and therefore \( m \in (x \wedge y)(M) \). This shows that \( D \) is a commutative, idempotent semigroup.

In \( \mathcal{C}(\mathcal{L}) \) a theory is more informative than an other one, if the former contains the latter. So the information order in \( \mathcal{C}(\mathcal{L}) \) is \( S \leq T \) if \( S \subseteq T \). Therefore, combination is the join in the complete lattice \( \mathcal{C}(\mathcal{L}) \), that is,

\[
S \vee T = C_{\models}(S \cup T).
\]
Here the bottom element is given by the tautologies \( C_{\bot}(\emptyset) \) and the top element \( C_{\top}(\mathcal{L}) = \mathcal{L} \), which as usual is to be considered as expressing contradiction. Further, for any \( x \in D \) and \( S \in C(\mathcal{L}) \), we define

\[
\mathbf{x}(S) = \mathbf{r}(\mathbf{x}(\mathbf{r}(S))).
\]

We leave it to the reader to verify that \((C(\mathcal{L}), D)\) is an information algebra with this definition of operators \( x \in D \). In fact it is isomorph to the algebra \((C(\mathcal{M}), D)\), as will be shown in Section 5.2, see also (Kohlas & Schneuwly, 2009).

There are many concrete instances of such systems: For example, consider for \( \mathcal{L} \) a propositional language over a countable set \( p_1, p_2, \ldots \) of propositional variables, let \( \mathcal{M} \) be the set of all valuations and \( \models \) the usual satisfying relation of propositional logic. For any finite subset \( I \) of \( \omega = \{1, 2, \ldots\} \), consider the equivalence relation \( v \equiv_I w \) if, \( v(i) = w(i) \) for all \( i \in I \) between valuations. This family of equivalence relations satisfies all requirements above relative to the lattice of finite subsets of \( \omega \). The resulting information algebra is closely related to the Lindenbaum algebra, but with a family of existential operators in addition. Similarly information algebras of structures and theories of predicate logic may be obtained (Henkin et al., 1971; Kohlas & Schneuwly, 2009). Further we may consider linear equations or linear inequalities over finite sets of variables and solutions in the corresponding vector spaces. This leads to information algebras of affine spaces or convex polyhedra respectively as mentioned in the previous section (Kohlas, 2003a).

We remark that the information algebra \((C(\mathcal{L}, D)\) can also be considered as an information algebra of an information system. To each \( x \in D \) associate the sublanguage

\[
L_x = \{ s \in \mathcal{L} : \mathbf{r}(\{s\}) = x(\mathbf{r}(\{s\})) \}
\]

of \( \mathcal{L} \). With each of these sublanguages associate the operators

\[
C_x(C_{\models}(S)) = C_{\models}(C_{\models}(S) \cap L_x).
\]

Now, it is easy to verify that \( L_x \cap L_y = L_{x \land y} \). Further, if \( x \neq y \) implies \( \equiv_x \neq \equiv_y \), then the mapping \( x \mapsto L_x \) is one-to-one and so is \( x \mapsto C_x \). Since the pair of maps \( M \mapsto \mathbf{r}(M) \) and \( x \mapsto C_x \) form an information algebra isomorphism, we obtain

\[
\mathbf{r}(x(M)) = C_x(\mathbf{r}(M)).
\]

This shows that indeed the information algebra \((C(\mathcal{L}, C(D))\), where \( C(D) \) denotes the family of all operators \( C_x \) for \( x \in D \), is an algebra of an information system (see Section 4.2)
4.4 Labeled Algebras

Here we introduce a related, but different algebraic structure describing information, which has many models and always induces an information algebra. It is also a structure important for computational purposes (Kohlas & Shenoy, 2000). We start by presenting a motivating example.

Example 4.2 Relational Algebra. An important structure for information algebras is the relational algebra of database theory: Let \( I \) be a set of variables or attributes. For each \( i \in I \) let \( U_i \) be the non-empty set of possible values of variable \( i \). Let \( s \) be a subset of \( I \). An \( s \)-tuple is a function \( f \) with domain \( s \) and \( f(i) \in U_i \) for all \( i \in s \). The set of all \( s \)-tuples is denoted by \( E_s \). For any \( s \)-tuple and any subset \( t \subseteq s \), the restriction of \( f \) to \( t \) is denoted by \( f[t] \). Note that this is in fact a functional which assigns to a tuple \( f \) a new tuple \( f[t] \). A relation \( R \) over a set \( s \) is a set of \( s \)-tuples. The set \( s \) is called the domain of \( R \) and denoted by \( d(R) \). For \( t \subseteq s \), the (relational) projection of \( R \) to \( t \) is defined as follows:

\[
\pi_t(R) = \{ f[t] : f \in R \}.
\]

The (relational) join of a relation \( R \) over \( s \) and a relation \( S \) over \( t \) is

\[
R \bowtie S = \{ f \in E_{s\cup t} : f[s] \in R, f[t] \in S \}.
\]

Note that \( E_\emptyset = \{ \{ \emptyset \} \} \), and that \( f[\emptyset] = \{ \emptyset \} \), \( \pi_\emptyset(R) = E_\emptyset \) if \( R \neq \emptyset \) and \( \pi_\emptyset(\emptyset) = \emptyset \).

\[
E = \bigcup_{s \subseteq I} E_s.
\]

be the set of all relations over any subset of variables.

For the sequel we note the following:

1. Each relation \( R \) pertains to a certain \( s \)-tuple of attributes. We call this the label of the relation and denote it by \( d(R) = s \). Then \( R \) can be seen as an information relative to the set \( s \) of variables, it indicates possible values of these variables.

2. The relational join is a binary operation, which combines two pieces of information \( R \) and \( S \) into a new one, \( R \bowtie S \).

3. The relation projection extracts from a relation \( R \) relative to a set \( s \) of attributes, \( d(R) = s \), the part \( \pi_t(R) \) relative to a subset \( t \subseteq s \) of attributes.

We note the following well-known properties relative to these operations:

1. The family \( D \) of subsets of \( I \) is a lattice.
2. The set \( R \) of relations over subsets of \( I \) is under join a commutative semigroup. For any element \( s \) of \( D \) there is the full relation \( E_s \) of all \( s \)-tuples, which is the neutral element of relational join with respect to relations over \( s \), \( R \bowtie E_s = R \) if \( d(R) = s \). The empty relation \( Z_s = \emptyset \), is the null element for the relational join, \( R \bowtie Z_s = Z_s \) if \( d(R) = s \).

3. The label of a relational join is the union of the labels of the factors, \( d(R \bowtie S) = d(R) \cup d(S) \).

4. The label of a projection of a relation to some subset \( t \) of attributes equals \( t \), \( d(\pi_t(R)) = t \).

5. Projection of a relation may be executed stepwise, \( \pi_t(\pi_s(R)) = \pi_t(R) \) if \( t \subseteq s \subseteq d(R) \).

6. The semijoin identity holds, that is, \( \pi_s(R \bowtie S) = R \bowtie (\pi_s \cap t)(S) \) if \( d(R) = s \) and \( d(S) = t \).

7. Relational join is idempotent in the following sense: \( R \bowtie \pi_t(R) = R \).

8. A projection of the full relation yields again a full relation, \( \pi_t(E_s) = E_t \).

9. A relational join of an empty relation with the full relation on a larger set of attributes remains empty, \( Z_t \bowtie E_s = Z_s \) if \( t \subseteq s \).

Of course in a relation algebra other operations such as unions, differences, and so on might be considered. But for our purposes relational join and projection are the essential ones.

The properties of a relational algebra exhibit a structure of some interest, which we shall now generalize. We assume this time \( D \) to be lattice, not necessarily distributive, and its elements \( x \) represent domains of information. This is like the subsets of variables \( s \) in the relational algebra, which identify domains to which relations as pieces of information refer. We may also see these labels \( s \) as representing questions like, what are the possible values of the attributes in \( s \)? The lattice operations of \( D \) correspond then to the formation of new question. So \( s \cup t \) is the combined question of \( s \) and \( t \), that answers both question relative to \( s \) and \( t \). Further, \( s \cap t \) is the common part of questions \( s \) and \( t \). Finally \( t \subseteq s \) means that \( t \) represent a coarser question than \( s \) or \( s \) a finer one than \( t \). In this sense, lattice \( D \) might also be seen as a system of related questions.

There is further a semigroup \( \Psi \), whose elements represent pieces of information and the semigroup operation represents combination or aggregation of information. This corresponds to relational join in the relational algebra. Each element \( \psi \in \Psi \) pertains to a specific domain in \( D \) which is denoted by \( d(\psi) \). This expresses the idea that any piece of information relates to
an identified question. Finally there is an information-extraction operation
which for any domain \( y \leq d(\psi) \) extracts the part of information of \( \psi \) pertaining to \( y \) and denoted by \( \psi^\downarrow_y \). In relational algebra this is relational projection.

Formally, we then have the operations

1. Labeling: \( d : \Psi \rightarrow D, \psi \mapsto d(\psi) \),
2. Combination: \( \otimes : \Psi \times \Psi \rightarrow \Psi, (\phi, \psi) \mapsto \phi \otimes \psi \),
3. Projection: \( \downarrow : \Psi \times D \rightarrow \Psi, (\psi, y) \mapsto \psi^\downarrow_y \), defined for \( y \leq d(\psi) \).

We require that the following conditions or axioms are satisfied, following
the model of relational algebra:

1. \( D \) a lattice.
2. \( \Psi \) is a commutative semigroup, and \( \forall y \in D \) there is an element \( 0_y \) and an element \( 1_y \), with \( d(0_y) = d(1_y) = y \) and such that \( \forall \psi \in \Psi \) with \( d(\psi) = y \), \( 0_y \otimes \psi = \psi \) and \( 1_y \otimes \psi = 1_y \)
3. \( \forall \phi, \psi \in \Psi, d(\phi \otimes \psi) = d(\phi) \lor d(\psi) \).
4. \( \forall \psi \in \Psi, \forall y \in D \) such that \( y \leq d(\psi) \), \( d(\psi^\downarrow_y) = y \).
5. \( \forall \psi \in \Psi, \forall x, y \in D \) such that \( x \leq y \leq d(\psi) \), \( (\psi^\downarrow_y)^\downarrow_x = \psi^\downarrow_{y \land x} \).
6. \( \forall \phi, \psi \in \Psi \) such that \( d(\phi) = x \), \( d(\psi) = y \), \( (\phi \otimes \psi)^\downarrow_x = \phi \otimes \psi^\downarrow_{x \land y} \).
7. \( \forall \psi \in \Psi, \forall y \in D \) such that \( y \leq d(\psi) \), \( \psi \otimes \psi^\downarrow_y = \psi \).
8. If \( x \leq y \), then \( 0_y^\downarrow_x = 0_x \).
9. If \( x \leq y \), then \( 0_y \otimes 1_x = 1_y \).

Such a system \( (\Psi, D) \) is called a labeled information algebra.

A somewhat weaker version of this system, in particular without condition 7, has originally been proposed by (Shenoy & Shafer, 1990) as an
axiomatic base for a generic computational mechanism called local computation
unifying several seemingly different algorithms from quite different
fields of Computer Science (Kohlas & Shenoy, 2000; Kohlas, 2003a).

These axioms have the following informal justification: Pieces of information can be combined in any order, the \( 0_x \)-element represents the vacuous
information relative to \( x \), it changes no other information pertaining to the
same domain. The \( 1_x \) models contradiction, it destroys any other information.
Item 3 says that a combined information pertains to the combined
questions of the individual factors. Item 6 is the generalization of the semi-join property of relational algebra. It is the most important axiom, both for
theoretical as well as for computational purposes. It resembles the property
of existential quantification in information algebra and in fact plays a similar role. Item 7 says that combination a piece of information with itself or part of itself gives nothing new. So, these are reasonable requirement for a model of information. And they somehow resemble those of an information algebra, without being identical.

Here are a few immediate consequences: We may define a partial order in $\Psi$ by $\phi \leq \psi$ if $\phi \otimes \psi = \psi$. Then, in this order, $\phi \otimes \psi = \sup \{\phi, \psi\}$ and $\Psi$ becomes a join semilattice. If $\Psi_y$ denotes the subset of $\Psi$ of all elements with domain $y$, $d(\psi) = y$, then $\Psi_y$ is itself a semilattice with $0_y$ and $1_y$ as least and greatest element (Kohlas, 2003a). The element $0_y$ represents vacuous information “pertaining” to domain $y$ and the element $1_y$ represents the contradiction within domain $y$. Further, if $x \leq y$, then $1_y^x = 1_x$, for arbitrary $x, y \in D$, and $1_x \otimes 1_y = 1_x \lor y$. We refer to (Kohlas, 2003a) for an in depth analysis of labeled information algebras.

In a labeled information algebra $(\Psi, D)$, each piece of information is associated with its domain $d(\psi)$. But it can be transported to another domain, using projection. For this purpose, note that if $d(\psi) = x$ and $x \leq y$, the combined information $\psi \otimes 0_y$ pertains to domain $y$, and $(\psi \otimes 0_y)^{\uparrow y} = \psi$. So

$$\psi^{\uparrow y} = \psi \otimes 0_y$$

(4.5)

can be considered as the vacuous extension of $\psi$ to a larger domain $y$, vacuous, because it adds no information. We add here for later reference a number of elementary properties of this operation:

**Lemma 4.3**

1. If $x \leq y$, then $0_y^\uparrow x = 0_y$, $1_x^\uparrow y = 1_y$.

2. If $d(\phi) = x$, then $\phi \otimes 0_y = \phi^{\uparrow x \lor y}$.

3. If $d(\phi) = x$, then $\phi^{\uparrow x} = \phi$.

4. If $d(\phi) = x$, and $x \leq y \leq z$, then $(\phi^{\uparrow y})^{\uparrow z} = \phi^{\uparrow z}$.

5. If $d(\phi) = x$, $d(\psi) = y$, and $x, y \leq z$, then $(\phi \otimes \psi)^{\uparrow z} = \phi^{\uparrow z} \otimes \psi^{\uparrow z}$.

6. If $d(\phi) = x$, $d(\psi) = y$, then $\phi \otimes \psi = \phi^{\uparrow x \lor y} \otimes \psi^{\uparrow x \lor y}$.

7. If $d(\phi) = x$, and $x \leq y$, then $(\phi^{\uparrow x})^{\downarrow y} = \phi$.

For the elementary proof of these properties see (Kohlas, 2003a).

Next we define the transport operator for any $x, y \in D$ and $\psi \in \Psi$ such that $d(\psi) = x$,

$$\psi^{\rightarrow y} = (\psi^{\uparrow x \lor y})^{\downarrow y}.$$  

(4.6)

It can also be shown that alternatively, (Kohlas, 2003a),

$$\psi^{\rightarrow y} = (\psi^{\downarrow x \land y})^{\uparrow y}.$$
Again, we add a number of properties of the transport operation for later reference:

**Lemma 4.4**  
1. If \( d(\phi) = x \), then \( \phi \rightarrow x = \phi \).
2. If \( d(\phi) = x \), and \( x, y \leq z \), then \( \phi \rightarrow y = (\phi \uparrow z) \downarrow y \).
3. If \( d(\phi) = x \), and \( x \leq y \), then \( \phi \rightarrow y = \phi \uparrow y \).
4. If \( d(\phi) = x \), and \( y \leq x \), then \( \phi \rightarrow y = \phi \downarrow y \).
5. If \( y \leq z \), then \( \phi \rightarrow y = (\phi \rightarrow z) \rightarrow y \).
6. \( (\phi \rightarrow x) \rightarrow y = (\phi \rightarrow x \wedge y) \rightarrow y \) for any \( x, y \).
7. If \( d(\phi) = x \), then \( (\phi \otimes \psi) \rightarrow x = \phi \otimes \psi \rightarrow x \).

For the proof of these properties see (Kohlas, 2003a).

If, for \( \phi, \psi \in \Psi \) with \( d(\phi) = x \) and \( d(\psi) = y \) we have \( \phi \rightarrow y = \psi \) and \( \psi \rightarrow x = \phi \), then we may consider that the two elements \( \phi \) and \( \psi \) represent the same information. Therefore we define, if \( d(\phi) = x \) and \( d(\psi) = y \),

\[
\phi \equiv \sigma \psi \text{ if } \phi \rightarrow y = \psi \text{ and } \psi \rightarrow x = \phi.
\]

This is a congruence relative to the operation of combination and transport (Kohlas, 2003a). Therefore, within the set \( \Psi / \equiv \) we may unambiguously define

\[
[\phi]_\sigma \vee [\psi]_\sigma = [\phi \otimes \psi]_\sigma, \quad x([\psi]_\sigma) = [\psi \rightarrow x]_\sigma,
\]

where \([\phi]_\sigma\) denotes the equivalence class of \( \phi \) in the congruence \( \sigma \). If we define \([\phi]_\sigma \leq [\psi]_\sigma\) if \([\phi]_\sigma \otimes [\psi]_\sigma = [\psi]_\sigma\), then \( \Psi / \equiv \) becomes a join semilattice with \([\phi]_\sigma \vee [\psi]_\sigma = [\phi]_\sigma \otimes [\psi]_\sigma\) and it can be verified that \( (\Psi / \equiv, D) \) is an information algebra. Its 0-element is the equivalence class \([0]_\sigma\) and its 1-element is the equivalence class \([1]_\sigma\). In this way, any labeled information algebra induces an information algebra. Note that this algebra is supported, since \( x([\phi]_\sigma) = [\phi \rightarrow x]_\sigma = [\phi]_\sigma\) if \( d(\phi) = x \). We state this as a theorem, whose detailed proof may be found in (Kohlas, 2003a)

**Theorem 4.1** Let \( (\Psi, D) \) be a labeled information algebra. Then \( (\Psi / \equiv, D) \) is a supported information algebra.

This shows that via labeled information algebra new information algebras may be introduced. We shall use this later several times.

We remark that not only an information algebra can be derived from a labeled information algebra, but inversely, a labeled information algebra
can be associated with any supported information algebra \((\Phi, D)\), provided \(D\) is a lattice. Let \(\Psi_x = \{(\phi, x) : \phi = x(\phi)\}\) and

\[
\Psi = \bigcup_{x \in D} \Psi_x.
\]

Define then \(d(\phi, x) = x\) and for \((\phi, x)\) and \((\psi, y)\) in \(\Psi\),

\[
(\phi, x) \otimes (\psi, y) = (\phi \vee \psi, x \vee y),
\]

This is justified by property 7. of support in Section 2.1. Further, if \(y \leq x\),

\[
(\phi, x)^\downarrow y = (y(\phi), y).
\]

It can be verified that \((\Psi, D)\) is a labeled information algebra (Kohlas, 2003a):

**Theorem 4.2** Let \((\Phi, D)\) be a supported information algebra and \(D\) a lattice. Then \((\Psi, D)\) with \(\Psi = \{(\psi, x) : \psi = x(\psi)\}\) with combination and projection defined by (4.7) and (4.8) respectively, is a labeled information algebra.

We remark also that \((\Psi/\equiv_{\sigma}, D)\) is isomorphic to \((\Phi, D)\) under the isomorphism \(\psi \mapsto ([\psi]_{\sigma}, x)\) (Kohlas, 2003a).

We noted in Section 2.1 that in an information algebra \((\phi, D)\) we may adjoin a top element to the semilattice \(D\) as the identity map \(id : \Phi \to \Phi\) and then \((\Phi, D \cup \{id\})\) is still an information algebra, even a supported one. Its associated labeled algebra \((\Psi, D \cup \{id\})\) differs from \((\Psi, D)\) simply by a new domain \(\Psi_{id} = \{(phi, id) : \phi \in \Phi\}\) added to the old domains \(\Psi_x\) for \(x \in D\). Otherwise anything remains as it was. We may therefore ask, whether, conversely, starting with a labeled information algebra \((\Psi, D)\), how we may add a new top domain \(\top \in D\) if \(D\) does not already have one. This is in fact possible. As well-known, \(D \cup \{\top\}\) is still a lattice if \(D\) is one, with \(x \land \top = x\) and \(x \lor \top = \top\). Then, of course, \(x \leq \top\) for all \(x \in D\). Further, we add a new top domain \(\Psi_\top\) consisting of elements \(\psi_\top\) such that \(\psi_\top = \phi_\top\) if \(\psi \equiv_{\sigma} \phi\). That is, for each class of equivalent labeled information we add exactly a new element in the domain \(\Psi_\top\), such that

\[
\Psi_\top = \{\psi_\top : [\psi]_{\sigma} \in \Psi/\sigma\}.
\]

We then extend the operations of the labeled information algebra \((\Psi, D)\) to the extended system \((\Psi \cup \Psi_\top, D \cup \{\top\})\) in the following way:

1. **Labeling**: \(d(\psi_{\top}) = \top\) for all \(\psi_\top \in \Psi_\top\).
2. **Combination**: \(\phi \otimes \psi_\top = (\phi \otimes \psi)_\top\) for all \(\phi \in \Psi \cup \Psi_\top\) and \(\psi_\top \in \Psi_\top\).
4.4. LABELED ALGEBRAS

3. **Projection:** \( \psi_{\top}^x = \psi^{\rightarrow x} \) for all \( \psi_{\top} \in \Psi_{\top} \) and \( x \in D \). Further \( \psi_{\top}^\top = \psi_{\top} \).

These operations are all well-defined. Labeling is no problem anyway. As for combination, if \( \psi \equiv_{\sigma} \psi' \), then \( \phi \otimes \psi \equiv_{\sigma} \psi' \), since \( \equiv_{\sigma} \) is a congruence as stated above. So, we have \( \phi \otimes \psi_{\top} = (\phi \otimes \psi)_{\top} = (\phi \otimes \psi')_{\top} = \phi \otimes \psi'_{\top} \).

For projection similarly, if \( \psi \equiv_{\sigma} \psi' \), and \( d(\psi) = y \), \( d(\psi') = z \), we have

\[
\psi_{\top}^{\text{↓}x} = \psi_{\top}^{\rightarrow x} = (\psi_{\top}^{\text{↓}y^{\top}z})^x = (\psi_{\top}^{\text{↓}y^{\top}z})^{\text{↓}x} = \psi_{\top}^{\text{↓}x}.
\]

We claim that with these definitions, \( (\Psi \cup \Psi_{\top}, D \cup \{\top\}) \) becomes a labeled information algebra.

**Theorem 4.3** Let \( (\Psi, D) \) be a labeled information algebra, where \( D \) has no top element. Then \( (\Psi \cup \Psi_{\top}, D \cup \{\top\}) \) with the operations as defined above is a labeled information algebra.

**Proof.** Note that the axioms of a labeled information must be verified only as far as elements of \( \Psi_{\top} \) are involved. So, in axiom 2 we must, for instance verify that \( (\phi \otimes \psi) \otimes \eta_{\top} = \phi \otimes (\psi \otimes \eta_{\top}) \). This holds, since, invoking associativity in \( \Psi \),

\[
(\phi \otimes \psi) \otimes \eta_{\top} = ((\phi \otimes \psi) \otimes \eta)_{\top} = (\phi \otimes (\psi \otimes \eta))_{\top} = \phi \otimes (\psi \otimes \eta_{\top}.
\]

We leave it to the reader to verify the other cases of associativity. That commutativity holds is evident from the definition of combination. The neutral element in \( \Psi_{\top} \) is \( 0_{\top} \) and contradiction is represented by the element \( 1_{\top} \).

Axiom 3 holds, since \( d(\phi \otimes \psi_{\top}) = d((\phi \otimes \psi)_{\top}) = \top \), which equals \( d(\phi) \lor d(\psi_{\top}) = d(\phi) \lor \top = \top \). Further, by definition \( d(\psi_{\top}^x) = d(\psi^{\rightarrow x}) = x \), and so axiom 4 holds too.

If \( x \leq y \), then

\[
(\psi_{\top}^y)^x = (\psi^{\rightarrow y})^x = \psi^{\rightarrow x \land y} = \psi^{\rightarrow x} = \psi_{\top}^x.
\]

We refer to Lemma 4.4 for the properties of the transport operation used here. This is axiom 5.

Assume \( d(\phi) = x \). Then

\[
(\phi \otimes \psi_{\top})^x = (\phi \otimes \psi)^{\top x} = (\phi \otimes \psi)^{\rightarrow x} = \phi \otimes \psi^{\rightarrow x} = \phi \otimes \psi_{\top}^{\rightarrow x},
\]

see Lemma 4.4. This proves axiom 6. Also, \( \psi_{\top} \otimes \psi_{\top}^\top = \psi_{\top} \otimes \psi_{\top}^{\rightarrow x} = (\psi \otimes \psi^{\rightarrow x})_{\top} = \psi_{\top} \), which shows that axiom 7 holds too.
Axioms 8 and 9 are proved by $0_\top^x = 0_x^{-x} = 0_x$, and $0_x \otimes 1_\top = (0_x \otimes 1_x)_\top = 1_\top$.

This concludes the proof. \hfill $\square$

Clearly, $(\Psi, D)$ is a subalgebra of $(\Psi \cup \Psi_\top, D \cup \{\top\})$. Further, in $\Psi \cup \Psi_\top$ we have $\psi \equiv_\top \psi_\top$ and the equivalence classes $[\psi]_\top$ in $\Psi \cup \Psi_\top$ are the same as those in $\Psi$, except that the element $\psi_\top$ is added to the class. Therefore, the information algebras $(\Psi / \sigma, D \cup \{\top\})$ and $(\Psi \cup \Psi_\top / \sigma, D \cup \{\top\})$ associated with the labeled information algebras $(\Psi, D)$ and $(\Psi \cup \Psi_\top, D \cup \{\top\})$ are essentially identical.

**Example 4.3 Convex Sets:** In Section 3.4 convex sets were mentioned as an example of an information algebra. There is a labeled version of this algebra: Let $D$ be the lattice of finite subsets of $\omega = \{0, 1, 2, \ldots\}$. It is in fact a relational algebra as discussed in Example 4.2. Let $R_s$ be the vector space of vectors (or tuples) $v : s \to \mathbb{R}$, where $s$ belongs to $D$. The projection of a vector $v \in R_s$ to a subset $t$ is denoted by $v[t]$ and is the restriction of $v$ to $t$. Denote by $\Psi_s$ the set of convex subsets of $R_s$ and define

$$\Psi = \bigcup_{s \in D} \Psi_s$$

If $S \in \Psi_s$, then it has the label $d(S) = s$. If $S$ and $T$ are two convex sets, $d(S) = s$ and $d(T) = t$, then *combination* is defined by the relational join

$$S \otimes T = S \bowtie T = \{v \in R^{s \cup t} : v[s] \in S, v[t] \in T\}.$$ 

Further, *projection* of a convex set $S \in \Psi_s$ to $t \subseteq s$ is defined as

$$\pi_t(S) = \{v[t] : v \in S\}.$$ 

With these operation $\Psi, D$) forms a labeled information algebra $(\Psi, D)$, in fact a subalgebra of the relational algebra of tuples in $\mathbb{R}^s$, for finite sets $s$.

The cylindric extension of a set $S$ with $d(S) = s$ to $t \supseteq s$ is defined as $c_\omega(S) = \{v \in \mathbb{R}^t : v[s] \in S\}$. This corresponds to the vacuous extension in the labeled information algebra. Note that we may also consider the cylindric extension to $\omega$, $c_\omega(S)$. So we have $S \equiv_\sigma T$ if $c_\omega(S) = c_\omega(T)$, if $d(S) = s$ and $d(T) = t$. This is equivalent to $c_\omega(S) = c_\omega(T)$. We say that a subset $R$ of $\mathbb{R}^\omega$ has a base in $s$, if $R = c_\omega(S)$ for some $S \in \Psi_s$. Then we may see the information algebra $(\Psi / \sigma, D)$ associated with the labeled algebra $(\Psi, D)$ as the set algebra of subsets in $\mathbb{R}^\omega$ with base in $s \in D$. Combination is simply intersection and extraction is defined as $s(c_\omega(S)) = c_\omega(\pi_s c_\omega(S))$.

According to the discussion above, we may add the domain $\omega$ to $D$ and consider the lattice $D \cup \{\omega\}$. The elements of $\Psi_\omega$ are exactly the sets $c_\omega(S)$ for $S \in \Psi$. Combination is defined as $S \otimes c_\omega(T) = c_\omega(S \otimes T)$. This is essentially still a relational join.
Note that in this the labeled information elements on the top domain $\mathbb{R}^\omega$ correspond essentially to the cylindric sets considered in the associated information algebra $(\Psi/\sigma, D)$. Further, the labeled algebras $(\Psi, D)$ and $(\Psi \cup \Psi_T, D \cup \{\top\})$ generate essentially the same the same information algebras $(\Psi/\sigma, D)$, $(\Psi/\sigma, D \cup \{\top\})$ and $(\Psi \cup \Psi_T/\sigma, D \cup \{\top\})$.

The last remark in the example above applies more generally to any labeled information $(\Psi, D)$. Consider any labeled information algebra $(\Psi, D)$ where $D$ has a top element $\top$. If $D$ does not have a top element, then consider $(\Psi \cup \Psi_T, D \cup \{\top\})$. Then $\Psi_T$ is a semilattice with bottom element $0$ and top element $0\top$. Hence we may, for elements $\phi, \psi \in \Psi$, write $\phi \otimes \psi = \phi \lor \psi$. Define further for any $\phi \in /Psi$ and any $x \in D$,

$$\bar{x}(\phi) = (\phi^{lx})^{\top}. \quad (4.9)$$

The each $x \in D$ defines thereby a map $\bar{x} : \Psi_T \to \Psi_{\text{top}}$. Let $\tilde{D}$ be the set of these mappings for $x \in D$. Then, we claim that $(\Psi_T, \tilde{D})$ is an information algebra, and it is isomorphic to $(\Psi/\sigma, D)$.

**Theorem 4.4** If $(\Psi, D)$ is a labeled information algebra, where the lattice $D$ has a top element $\top$, then $(\Psi_T, \tilde{D})$ is an information algebra and $(\Psi_T, \tilde{D})$ is isomorphic to $(\Psi/\sigma, D)$.

**Proof.** We have already stated the $\Psi_T$ is a semilattice with bottom and top element. It remains to verify first that the maps $\bar{x}$ are existential quantifiers. First, it follows directly from (4.9) that $\bar{x}(1) = 1_T$. Also, by elementary properties of projection and vacuous extension (consult Lemma 4.3), we see that $\bar{x}(\psi)$ is a semilattice isomorphism between $\Psi_T$ and $D$. Then, we claim that $(\Psi_T, \tilde{D})$ is an information algebra.

Further, by properties stated in Lemma 4.4, we find also that

$$\bar{x}(\bar{y}(\psi)) = (((\psi^{ly})^{lx})^{\top})^{\top} = (((\psi^{ly})^{lx})^{\top\top})^{\top} = (((\psi^{ly})^{lx})^{\top\top})^{\top} = (((\psi^{ly})^{lx})^{\top\top})^{\top} = (((\psi^{ly})^{lx})^{\top\top})^{\top}.$$

The symmetry of the last formula shows that $\bar{x}(\bar{y}(\psi)) = \bar{y}(\bar{x}(\psi)) = x \bar{\land} y(\psi)$. This proves that $\tilde{D}$ is a commutative, idempotent semigroup of operators. So, $(\Psi_T, \tilde{D})$ is indeed an information algebra.

The map $x \mapsto \bar{x}$ is clearly a lattice isomorphism between $\tilde{D}$ and $D$. The map $\psi \mapsto [\psi]_{\sigma}$ is also obviously a semilattice isomorphism between $\Psi_T$ and
Ψ/σ. Finally, we have $[\bar{\xi}(\psi)]_\sigma = \left[\left(\psi^{\downarrow x}\right)^\top\right]_\sigma = [\psi^{\top x}]_\sigma = x([\psi]_\sigma)$. This shows that $(\Psi_\top, D)$ and $(\Psi/\sigma, D)$ are isomorphic. 

This completes the picture of the relation between labeled information algebras and information algebras.

Since ideal completion (see Section 2.2) is essentially an order-theoretic concept, it can also be applied to labeled information algebras $(\Psi, D)$. Here, we consider ideals in $\Psi_x$ for any domain $x$. Thus, we define:

**Definition 4.2 Labeled Ideals:** A non-empty set $I \subseteq \Psi_x$ is called an ideal of the domain $x$, if the following holds:

1. $\phi \in I$ and $\psi \in \Psi_x, \psi \leq \phi$ imply $\psi \in I$.
2. $\phi \in I$ and $\psi \in I$ imply $\phi \otimes \psi \in I$.

Again as before, an ideal defines informally a collection of consistent and complete set of pieces of information, this time with respect to a specified domain or question. An ideal $I$ in $\Psi_x$ is labeled with the domain $d(I) = x$. Let $I_{\Psi_x}$ be the set of ideals in $\Psi_x$. An ideal in $\Psi_x$ is called proper if it is different from $\Psi_x$. For $\phi \in \Psi_x$, the set $I(\phi) = \{\psi \in \Psi_x : \psi \leq \phi\}$ is an ideal, called a principal ideal. Define

$$I_{\Psi} = \bigcup_{x \in D} I_{\Psi_x}.$$  

Within $I_{\Psi}$ we define the operations of combination between ideals and projection of ideals according to the models of (2.2) and (2.3):

1. **Combination of Ideals:** Let $I_1 \in I_{\Psi_x}$ and $I_2 \in I_{\Psi_y}$. Then we define

$$I_1 \otimes I_2 = \{\psi \in \Psi_{x \vee y} : \psi \leq \phi_1 \otimes \phi_2 \text{ for some } \phi_1 \in I_1, \phi_2 \in I_2\}. \quad (4.11)$$

2. **Projection of Ideals:** Let $I \in I_{\Psi_y}$ and $x \leq y$. Then we define

$$I^{\downarrow x} = \{\psi \in \Psi_x : \psi \leq \phi^{\downarrow x} \text{ for some } \phi \in I\}. \quad (4.12)$$

It is easy to verify that both $I_1 \otimes I_2$ and $I^{\downarrow x}$ are indeed ideals. So the definitions above are well founded. Note also that $\phi^{\downarrow x} \in I^{\downarrow x}$, if $\phi \in I$.

As is to be expected, $(I_{\Psi}, D)$ with labeling, combination and projection as defined above is a labeled information algebra:

**Theorem 4.5** The algebraic structure $(I_{\Psi}, D)$ with labeling, combination and projection as defined above is a labeled information algebra.
4.4. LABELED ALGEBRAS

Proof. We are going to verify the axioms at the beginning of the section.

(1) Since \((\Psi, D)\) is a labeled information algebra, \(D\) is a lattice

(2) Semigroup: Commutativity of combination follows directly from the definition. To verify associativity we invoke labeling and obtain for \(I_1 \in I_{\Psi_x}\), \(I_2 \in I_{\Psi_y}\) and \(I_3 \in I_{\Psi_z}\)

\[
I_1 \otimes (I_2 \otimes I_3) = \{ \psi \in \Psi_x \otimes \Psi_y \otimes \Psi_z : \psi \leq \phi_1 \otimes \eta, \text{ for some } \phi_1 \in I_1, \eta \in I_2 \otimes I_3 \} = \{ \psi \in \Psi_x \otimes \Psi_y \otimes \Psi_z : \psi \leq \phi_1 \otimes \eta, \text{ for some } \phi_1 \in I_1, \eta \leq \phi_2 \otimes \phi_3 \text{ for some } \phi_2 \in I_2, \phi_3 \in I_3 \} = \{ \psi \in \Psi_x \otimes \Psi_y \otimes \Psi_z : \psi \leq \phi_1 \otimes \phi_2 \otimes \phi_3, \text{ for some } \phi_1 \in I_1, \phi_2 \in I_2, \phi_3 \in I_3 \}.
\]

The same result can also be derived for \((I_1 \otimes I_2) \otimes I_3\). This proves associativity of combination.

The principal ideal \(\{0_x\}\) is the neutral element of \(I_{\Psi_x}\), since for any \(I \in I_{\Psi_x}\),

\[
\{0_x\} \otimes I = \{ \psi \in \Psi_x : \psi \leq 0_x \otimes \phi = \phi \text{ for some } \phi \in I \} = I,
\]

and, for \(y \leq x\),

\[
\{0_x\}^{\downarrow y} = \{ \psi \in \Psi_y : \psi \leq 0_x^{\downarrow y} = 0_y \} = \{0_y\}.
\]

So axiom (8) holds too

The null element in \(I_{\Psi_x}\) is \(\Psi_x\). Certainly we have \(\Psi_x \otimes I = \Psi_x\) for any \(I \in I_{\Psi_x}\), since \(I \subseteq \Psi_x\). Further, if \(x \leq y\),

\[
\Psi_x \otimes \{0_y\} = \{ \psi \in \Psi_y : \psi \leq \phi \otimes 0_y = \phi^{\uparrow y} \text{ for some } \phi \in \Psi_x \} = \Psi_y,
\]

since for all \(\psi \in \Psi_y\) we have \(\psi \leq 1_x \otimes 0_y = 1_y\). This verifies also axiom (9).

(3) and (4) follow directly from the definition of combination and projection.

(5) Let \(x \leq y \leq z = d(I)\). Then, using axiom 6 in \((\Psi, D)\),

\[
(I^{\downarrow y})^{\uparrow x} = \{ \psi \in \Psi_y : \psi \leq \eta^{\uparrow y} \text{ for some } \eta \in I \}^{\downarrow x} = \{ \psi \in \Psi_x : \psi \leq \phi^{\uparrow x} \text{ with some } \phi \leq \eta^{\uparrow y} \text{ for some } \eta \in I \} = \{ \psi \in \Psi_x : \psi \leq \eta^{\uparrow x} \text{ for some } \eta \in I \} = I^{\uparrow x}.
\]

(6) Let \(I_1 \in I_{\Psi_x}\) and \(I_2 \in I_{\Psi_y}\). Then,

\[
(I_1 \otimes I_2)^{\downarrow x} = \{ \psi \in \Psi_x : \psi \leq \phi^{\downarrow x} \text{ for some } \phi \in I_1 \otimes I_2 \} = \{ \psi \in \Psi_x : \psi \leq \phi^{\downarrow x} \text{ with some } \phi \leq \phi_1 \otimes \phi_2 \text{ for some } \phi_1 \in I_1, \phi_2 \in I_2 \} = \{ \psi \in \Psi_x : \psi \leq (\phi_1 \otimes \phi_2)^{\downarrow x} = \phi_1 \otimes \phi_2^{\downarrow x} \land \eta \text{ for some } \phi_1 \in I_1, \phi_2 \in I_2 \} = \{ \psi \in \Psi_x : \psi \leq \phi_1 \otimes \eta_2 \text{ for some } \phi_1 \in I_1, \eta_2 \in I_2^{\downarrow x} \} = I_1 \otimes I_2^{\downarrow x} \land \eta_2.
\]
(7) Let $I \in \Psi_y$ and $x \leq y$. Then
\[ I \otimes I^{\downarrow x} = \{ \psi \in \Psi_y : \psi \leq \phi \otimes \eta \text{ form some } \phi, \eta \in I^{\downarrow x} \} \subseteq I, \]
since we may select $\eta = 0_y$. On the other hand certainly $I \subseteq I \otimes I^{\downarrow x}$, hence $I \otimes I^{\downarrow x} = I$.

Just as with information algebras, the labeled algebra $(\Psi, D)$ can be embedded into the labeled algebra $(I\Psi, D)$ by the mapping $\phi \mapsto I(\phi)$. We formulate the corresponding theorem, its proof is similar to the proof of Theorem 2.1 and is therefore omitted.

**Theorem 4.6** The labeled information algebra $(\Psi, D)$ can be embedded into the labeled information algebra $(I\Psi, D)$.

This justifies it again to call $(I\Psi, D)$ the **ideal completion** of the labeled algebra $(\Psi, D)$.

We leave it to the reader to examine the relation between ideal completions of associated labeled and information algebra.

### 4.5 Lattice Induced Algebras

Information algebras may be derived from any distributive lattice via labeled information algebras. Let $L$ be a **distributive lattice** with $0_L$ and $1_L$. Meet, join and order in $L$ are denoted by $\wedge_L$, $\vee_L$ and $\leq_L$ respectively. Consider any index set $I$ and an associated family of finite sets $\Omega_i$ for every $i \in I$.

For any finite subset $s$ of $I$ form the cartesian product $\Omega_s = \prod_{i \in s} \Omega_i$.

The elements of $\Omega_s$ are denoted by $x, y, \ldots$. These elements can also be considered as maps from $s$ into $\Omega_s$ such that $x(i) \in \Omega_i$. If $t \subseteq s$ and $x \in \Omega_s$, then $x[t]$ denotes the restriction of $x$ to $t$, the **projection** of $x$ to $t$. If $D$ is the distributive lattice of all finite subsets of $I$, then the system of all elements $x \in \Omega_s$ for all $s \in D$ becomes a **tuple system** (see Section 4.4) and the elements of $\Omega_s$ are called $s$-tuples.

**Lattice valuations** on $s$ are maps $\phi : \Omega_s \to L$. The set of all lattice valuations on $s$ is denoted by $\Phi_s$ and
\[ \Phi = \bigcup_{s \in D} \Phi_s \]
is the set of all valuations on finite subsets of $I$. Consider the following operations:
1. Labeling: \( d(\phi) = s \) if \( \phi \) is a lattice valuation on \( s \).

2. Combination: For \( \phi, \psi \in \Phi \) with \( d(\phi) = s \), \( d(\psi) = t \) the combination \( \phi \otimes \psi \) is defined for all \( x \in \Omega_{s,t} \) by

\[
(\phi \otimes \psi)(x) = \phi(x[s]) \land_L \psi(x[t]).
\]

3. Projection: For \( \phi \in \Phi \) with \( d(\phi) = s \) and \( t \subseteq s \), the projection \( \phi^{lt} \) is defined for all \( x \in \Omega_t \) by

\[
\phi^{lt}(x) = \lor_L \{ \phi(z) : z \in \Omega_s : z[t] = x \}.
\]

It can easily be verified that \((\Phi, D)\) is a labeled information algebra, see also (Kohlas & Wilson, 2006). The vacuous information on \( s \), the \( 0_s \) element is the lattice valuation \( e_s(x) = 1_L \) for all \( x \in \Omega_s \), and the \( 1_s \) element is the map \( z_s(x) = 0_L \). Note that the order \( \leq \) in \( \Phi \), is the opposite of the order \( \leq_L \) in the lattice \( L \): If \( d(\phi) = d(\psi) = s \), then

\[
\phi \leq \psi, \text{ if, and only if } \psi(x) \leq_L \phi(x) \forall x \in \Omega_s.
\]

One might wonder why in the definition of the combination the meet in \( L \) is used instead of the join, since the combination is the join in the information order The reason is that in fact the information order in \( \Phi \) is inverse to the natural order in \( L \) as exemplified by the following examples.

**Example 4.4 Two Illustrative Examples:** Consider first the Boolean lattice \( L = \{0, 1\} \) with the natural order \( 0 \leq 1 \). Here the lattice valuations \( \phi(x) \) for \( s \)-tuples \( x \) represent indicator functions. Combination corresponds to set intersection and projection to ordinary set projection. So the lattice induced labeled information algebra based on this Boolean lattice corresponds to a labeled subset algebra.

Secondly consider the lattice \([0, 1]\) with the natural order between numbers, meet is minimum, join is maximum. Lattice valuations here can be thought to represent fuzzy sets of \( s \)-tuples or possibility distributions over \( s \)-tuples. Then combination is defined as

\[
(\phi \otimes \psi)(x) = \min\{\phi(x[s]), \psi(x[t])\}
\]

and projection by

\[
\phi^{lt}(x) = \max\{\phi(z) : z \in \Omega_s : z[t] = x\}.
\]

Again combination corresponds to a widely used definition of fuzzy set intersection and projection to the usual fuzzy set projection.
In this lattice-induced algebra $\Phi$ inherits in fact most of the lattice structure from $L$ (see also Section 6.6): First, for any $s \in D$ clearly, $\Phi_s$ is a distributive lattice: In the information order $\phi \otimes \psi = \phi \land_L \psi$, that is,

$$(\phi \lor \psi)(x) = \phi(x) \land_L \psi(x) \text{ for all } x \in \Omega_s. \quad (4.13)$$

But the meet $\phi \land \psi$ can also be defined by

$$(\phi \land \psi)(x) = \phi(x) \lor_L \psi(x) \text{ for all } x \in \Omega_s. \quad (4.14)$$

The distributivity of $L$ is also inherited in the lattices $\Phi_s$. In addition, if $d(\phi) = d(\psi) = s$ and $t \subseteq s$, then it can be verified that

$$(\phi \land \psi)^t(x) = \phi^t(x) \lor_L \psi^t(x) \text{ for all } x \in \Omega_s,$$

and if $t \supseteq s$, then

$$(\phi \land \psi)^t(x) = \phi^t(x) \land_L \psi^t(x) \text{ for all } x \in \Omega_s.$$ 

Therefore, the identities

$$(\phi \land \psi)^t = \phi^t \land \psi^t, \quad (\phi \land \psi)^t = \phi^t \land \psi^t$$

hold, if $d(\phi) = d(\psi) = s$.

If $d(\phi) = s$ and $d(\psi) = t$, then one can define, generalizing (4.14),

$$(\phi \land \psi) = \phi^s \land \psi^s \land \phi^t \land \psi^t$$

(4.15)

Note that (4.16) reduces to (4.14) if $s = t$. But (4.16) is however not the infimum in $\Phi$, since $\phi \land \psi \not\leq \phi, \psi$, and $\Phi$ as a whole is not a lattice. Nevertheless, $\equiv_\sigma$ is a congruence relative to the operation defined in $\Phi$ by (4.16), and therefore in $\Phi/\equiv_\sigma$ the meet can be defined by

$$[\phi]_{\sigma} \land [\psi]_{\sigma} = [\phi \land \psi]_{\sigma},$$

and it is indeed the infimum of $[\phi]_{\sigma}$ and $[\psi]_{\sigma}$. Then it can be verified that $\Phi/\equiv_\sigma$ becomes a distributive lattice and that for all $x \in D$,

$$x([\phi]_{\sigma} \land [\psi]_{\sigma}) = x([\phi]_{\sigma}) \land x([\psi]_{\sigma}).$$

So, in this case the information algebra $(\Phi/\equiv_\sigma, D)$ has quite some additional structure with respect to the simple information algebra structure:

1. $\Phi/\equiv_\sigma$ is a distributive lattice,

2. the identity (4.15) holds.

So this is a way to obtain lattice information algebras. In addition, if the lattice $L$ is a Boolean algebra, then $\Phi/\equiv_\sigma$ becomes also Boolean, since $\equiv_\sigma$ is also a congruence relative to complementation. Such lattice or Boolean information algebras are important special cases. For instance, the information algebras derived from cylindric algebras (see Section 4.1) are Boolean and the subset algebras (Section 3.3) are Boolean too.
4.6 Morphisms

Consider two information algebras \((\Phi_1, D_1)\) and \((\Phi_2, D_2)\) and order preserving maps between the semilattices \(\Phi_1\) and \(\Phi_2\). They provide another way to obtain new information algebras. Let \([\Phi_1 \rightarrow \Phi_2]\) be the set of order preserving maps \(f : \Phi_1 \rightarrow \Phi_2\) between the two semilattices \(\Phi_1\) and \(\Phi_2\). In \([\Phi_1 \rightarrow \Phi_2]\) we define a join \(f \lor g\) between order-preserving maps by the pointwise join in \(\Phi_2\):

\[(f \lor g)(\phi) = f(\phi) \lor_2 g(\phi).\]

In this way \([\Phi_1 \rightarrow \Phi_2]\) becomes a join-semilattice. The map \(\phi \mapsto 0_2\) is the 0 element and the map \(\phi \mapsto 1_2\) is the 1 element of this semilattice. On the other hand, \(D_1 \times D_2\) is a meet-semilattice under component-wise meet: \((x_1, x_2) \land (y_1, y_2) = (x_1 \land_1 y_1, x_2 \land_2 y_2)\). With any element \((x_1, x_2)\) in \(D_1 \times D_2\) an operator \((x_1, x_2) : [\Phi_1 \rightarrow \Phi_2] \rightarrow [\Phi_1 \rightarrow \Phi_2]\) can be associated in the following way: The map \((x_1, x_2)f\) between \(\Phi_1\) and \(\Phi_2\) is defined

\[((x_1, x_2)f)(\phi) = x_2(f(x_1(\phi))).\]

This is clearly an order preserving map, hence \((x_1, x_2)f \in [\Phi_1 \rightarrow \Phi_2]\). It can readily be checked that with these operator, \(([\Phi_1 \rightarrow \Phi_2], D_1 \times D_2)\) forms an information algebra (Kohlas, 2003a).

Such algebra are important and will be considered later in Sections 6.7 and 6.8.
Chapter 5

Subalgebras and Homomorphisms

5.1 Subalgebras

Here we start with elements of the algebraic theory of information algebras. Let \((\Phi, D)\) be an information algebra. Consider a pair \((\Psi, E)\), where

1. \(\Psi \subseteq \Phi\) is a sub-semilattice of \(\Phi\), containing \(0, 1\),
2. \(E \subseteq D\) is a sub-semilattice of \(D\),
3. \(\forall x \in E, x\) maps \(\Psi\) into \(\Psi\).

Then \((\Psi, E)\) is called a sub-information algebra, or shortly, when no confusion is possible, a sub-algebra, of \((\Phi, D)\). It is of course still an information algebra.

Here follow some generic examples of sub-algebras for later reference.

**Example 5.1** Here we hint at a few very simple sub-algebras of any information algebra \((\Phi, D)\) or \((\Phi, D \cup \{id\})\). So, \((\{0, 1\}, \{id\})\) is the simplest sub-algebra of \((\Phi, D \cup \{id\})\). Further, if \(\phi \in \Phi\) and \(x \in d\) such that \(x\) is a support of \(\Phi\), that is \(\phi = x(\phi)\), then \((\{0, \phi, 1\}, \{x\})\) and \((\{0, \phi, 1\}, \{x, id\})\) are subalgebras of \((\Phi, D)\) and \((\Phi, D \cup \{id\})\) respectively.

**Example 5.2** Suppose that \((\Phi, D)\) is an information algebra and \(D\) a lattice. Any sub-lattice of \(E\) of \(D\) induces a sub-algebra of \((\Phi, D)\) in the following way: Let \(\Phi_E\) be the subset of information elements supported by some element of \(E\)

\[\Phi_E = \{\phi \in \Phi : \exists x \in E \text{ such that } x(\phi) = \phi\}.\]

Then \(\Phi_E\) is a sub-semilattice of \(\Psi\). In fact, if \(\phi, \psi \in \Phi_E\), then there are \(x, y \in E\) such that \(\phi = x(\phi)\) and \(\psi = y(\psi)\). But then \(\phi \lor \psi = (x \lor y)(\phi \lor \psi) \in E\),
hence \( \phi \lor \psi \in \Phi_E \). Also \( 0,1 \in \Phi_E \). Thus \((\Phi_E, E)\) is a sub-algebra both of \((\Phi, D)\) and \((\Phi, D \cup \{id\})\) and \((\Phi_E, E \cup \{id\})\) is a sub-algebra of \((\Phi_D, D \cup \{id\})\).

**Example 5.3** In a similar spirit, as in the last example, let \( \Phi_x \subseteq \Phi \) be the subsets of information elements supported by \( x \),

\[
\Phi_x = \{ \phi \in \Phi : x(\phi) = \phi \}.
\]

Similarly, let \( \Phi_\phi \subseteq D \) be the set of domains supporting \( \phi \),

\[
\Phi_\phi = \{ x \in D : x(\phi) = \phi \}.
\]

Note that \( \Phi_x \) is a sub-semilattice containing \( 0,1 \) of \( \Phi \) (properties 1 and 6 of supports in Section 2.1) and that every \( y \in D \) maps \( \Phi_x \) into \( \Phi_x \) (property 2 of supports). Therefore, \((\Phi_x, D)\) is a sub-algebra of \((\Phi, D)\). Similarly, \( \Phi_\phi \) is a sub-semilattice of \( D \) (property 3 of supports), hence \((\Phi, \Phi_\phi)\) is also a sub-algebra of \((\Phi, D)\) and so is \((\Phi_x, \Phi_\phi)\).

In the models of information algebras of Chapter 3 many subalgebras can be found. Only a few are mentioned here: In the string example (Section 3.1) \((\Sigma_1^*, \omega)\) the string algebra \((\Sigma_2^*, \omega)\) is a subalgebra, if \( \Sigma_2 \subseteq \Sigma_1 \). In the subset algebra \((\Phi_D_1, D_1)\) associated with a lattice \( D_1 \) of partitions of an universe \( U \) \((\Phi_{D_2}, D_2)\) is a subalgebra if \( D_2 \subseteq D_1 \) is a sublattice of \( D_1 \) (Section 3.3). In the information algebra of convex sets \((C, \mathcal{P}(I))\) over subsets of variables (Section 3.4) the algebra of bounded convex sets is a subalgebra, so is the algebra of convex polyhedra. The algebra of convex polytopes (bounded polyhedra) is both a subalgebra of the algebra of bounded convex sets and of the one of the convex polyhedra.

### 5.2 Homomorphisms

Next we examine the concept of homomorphisms between two information algebras \((\Phi_1, D_1)\) and \((\Phi_2, D_2)\). Since an information algebra is two-sorted, we consider two maps

\[
f : \Phi_1 \to \Phi_2, \quad g : D_1 \to D_2.
\]

We require that \( f \) is a join-semilattice homomorphism maintaining join, \( 0 \) and \( 1 \), whereas \( g \) is a meet-semilattice homomorphism. In addition we require the following condition

\[
\forall \phi \in \Phi_1, \forall x \in D_1, \ f(x(\phi)) = g(x)(f(\phi)). \tag{5.1}
\]

If these conditions are satisfied, we say that the pair \((f, g)\) is a homomorphism between the information algebras \((\Phi_1, D_1)\) and \((\Phi_2, D_2)\). In case
$D_1 = D_2$, $g = \text{id}$ is possible, in which case (5.1) becomes $f(x(\phi)) = x(f(\phi))$ and in case $\Phi_1 = \Phi_2$ and $f = \text{id}$, (5.1) becomes $x(\phi) = g(x)(\phi)$.

Homomorphisms can be composed: Let $(\Phi_1, D_1)$, $(\Phi_2, D_2)$ and $(\Phi_3, D_3)$ be three information algebras and $(f_1, g_1)$ and $(f_2, g_2)$ be homomorphisms between $(\Phi_1, D_1)$, $(\Phi_2, D_2)$ and $(\Phi_2, D_3)$, $(\Phi_3, D_3)$ respectively. Then $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)$ is a homomorphism between $(\Phi_1, D_1)$ and $(\Phi_3, D_3)$. In fact, $f_2 \circ f_1$ and $g_2 \circ g_1$ are join- respectively meet homomorphisms between $\Phi_1$ and $\Phi_3$ and $D_1$ and $D_3$ respectively. Further, for $\phi \in \Phi_1$ and $x \in D_1$

\[
f_2 \circ f_1(x(\phi)) = f_2(f_1(x(\phi))) = f_2(g_1(x)(f_1(\phi)) = g_2(g_1(x)(f_2(f_1(\phi))) = g_2 \circ g_1(x)(f_2 \circ f_1(\phi))
\]

This shows that that $(f_2, g_2) \circ (f_1, g_1)$ is a homomorphism.

Next we introduce the kernel of a homomorphism $(f, g)$: Let $\phi \equiv f \psi$, if $f(\phi) = f(\psi)$ and $x \equiv_g y$ if $g(x) = g(y)$. We define then the kernel $(\Phi, D)/(\equiv_f, \equiv_g) = (\Phi/ \equiv_f, D/ \equiv_g)$ where $\Phi/ \equiv_f$ is the set of equivalence classes $[\phi]_f$ of the equivalence relation $\equiv_f$ and $D/ \equiv_g$ the set of equivalence classes $[x]_g$ of the equivalence relation $\equiv_g$. Clearly $\equiv_f$ and $\equiv_g$ are semilattice congruences. Therefore, we may define

\[
[\phi]_f \lor [\psi]_f = [\phi \lor \psi]_f, \quad [x]_g \land [y]_g = [x \land y]_g.
\]

By thereby, $\Phi_1/ \equiv_f$ becomes a join semilattice with $[0]_f$ and $[1]_f$ as 0 and 1, and $D_1/ \equiv_g$ a meet-semilattice. Moreover, if $\phi \equiv_f \psi$ and $x \equiv_g y$, then $f(x(\phi)) = g(x)f(\phi) = g(y)f(\psi) = f(y(\psi))$, hence by

\[
[x]_g[\phi]_f = [x(\phi)]_f
\]

an operator $[x]_g : \Phi_1/ \equiv_f \to \Phi_1/ \equiv_f$ is unambiguously defined. We claim that $(\Phi_1, D_1)/(\equiv_f, \equiv_g) = (\Phi_1/ \equiv_f, D_1/ \equiv_g)$ is an information algebra. It remains to verify conditions (a) to (c) for the operators $[x]_g$: Conditions (a) and (b) are evident. Condition (c) is verified by using the definition (5.2) and condition (c) in the information algebra $(\Phi_1, D_1)$,

\[
[x]_g([\phi]_f \lor [\psi]_f) = [x(x(\phi) \lor x(\psi))]_f = [x(\phi) \lor x(\psi)]_f = [x]_g([\phi]_f) \lor [x]_g([\psi]_f).
\]

Similarly, it can be verified in the same way, that

\[
[x]_g([y]_g([\phi]_f)) = [x(y(\phi))]_f = [(x \land y)(\phi))]_f = [x \land y]_g([\phi]_f).
\]

This shows that the operators form a commutative, idempotent semigroup.

The map $(i, j)$ defined by $i([\phi]_f) = f(\phi)$ and $j([x]_g) = g(x)$ is clearly an isomorphism between $(\Phi_1, D_1)/(\equiv_f, \equiv_g)$ and $(\Phi_2, D_2)$ and further $(f, g) = (i, j) \circ \pi$, where $\pi$ denotes the projection of $(\Phi_1, D_1)$ to $(\Phi_1, D_1)/(\equiv_f, \equiv_g)$.
defined by $\pi(\phi, x) = ([\phi]_f, [x]_g)$. This is the homomorphism theorem for information algebras.

As an illustration consider the information algebras $(C(M), D)$ and $(C(L), D)$ of model information and theories related to Galois connections (Section 4.3). Consider the mapping $M \mapsto f(M) = \hat{r}(M)$ from $C(M)$ to $C(L)$. This map is one-to-one since $\hat{r}(A) = \hat{r}(B)$ implies $A = \hat{r}(A) = \hat{r}(B) = B$ and it is onto $C(L)$ since $S = \hat{r}(\hat{r}(S))$ for any $S \in C(L)$, and $\hat{r}(S) \in C(M)$. Using well-known results of Galois connections, we see that for $M, N \in C(M)$,

$$\hat{r}(M \cap N) = \hat{r}(\hat{r}(M \cap N)) = \hat{r}(\hat{r}(M) \cap \hat{r}(N)) = \hat{r}(\hat{r}(M) \cup \hat{r}(N)).$$

This shows that the map $M \mapsto \hat{r}(M)$ is a semilattice homomorphism. In addition $M \mapsto \hat{r}(M) = C_\| (\emptyset)$, the bottom element in $C(L)$, and $\emptyset \mapsto \hat{r}(\emptyset) = L$, the top element. Finally, by definition of the operator $x \in D$ in $(C(L), D)$, $f(x(M)) = \hat{r}(x(M)) = \hat{r}(x(\hat{r}(M))) = x(\hat{r}(M)) = x(f(M))$. Thus, the map $(f, id)$ is a homomorphism, hence an isomorphism between $C(M)$ and $C(L)$.

### 5.3 Direct Products

Consider two information algebras $(\Phi_1, D_1)$ and $(\Phi_2, D_2)$ and construct the cartesian products $\Phi_1 \times \Phi_2$ and $D_1 \times D_2$. In the first product join is introduced pointwise and in the second meet is defined pointwise. In this way $\Phi_1 \times \Phi_2$ becomes a join-semilattice with $(0, 0)$ and $(1, 1)$ as 0 and 1 elements, and $D_1 \times D_2$ a meet-semilattice. For any element $x_1, x_2 \in D_1 \times D_2$ an operator

$$(x_1, x_2) : \Phi_1 \times \Phi_2 \rightarrow \Phi_1 \times \Phi_2$$

is defined by

$$(x_1, x_2)(\phi_1, \phi_2) = (x_1(\phi_1), x_2(\phi_2)).$$

We claim that in this way, $(\Phi_1 \times \Phi_2, D_1 \times D_2)$ becomes an information algebra. The only thing to check are the conditions (a) to (c) for the operators $(x_1, x_2) \in D_1 \times D_2$. Conditions (a) and (b) are evident. Condition (c) follows from the condition in each factor algebra individually since it holds pointwise.

The projections $(p_1, q_1)$, defined by $p_1(\phi_1, \phi_2) = \phi_1, q_1(x_1, x_2) = x_1$ and $(p_2, q_2)$, defined by $p_2(\phi_1, \phi_2) = \phi_2, q_2(x_1, x_2) = x_2$ are homomorphisms, since they maintain join, 0, 1, or meet and further $p_1((x_1, x_2)(\phi_1, \phi_2)) = x_1(\phi_1)$ and also $q_1(x_1, x_2)(p_1(\phi_1, \phi_2)) = x_1(\phi_1)$, and similar for the second projection. Consider now two homomorphisms $(f_i, g_i)$ from an information
5.3. DIRECT PRODUCTS

algebra \((\Phi, D)\) into \((\Phi_i, D_i)\), for \(i = 1, 2\). Define then \(f : \Phi_1 \times \Phi_2\) and \(g : D_1 \times D_2\) by

\[
f(\phi) = (f_1(\phi), f_2(\phi)), \quad g(x) = (g_1(x), g_2(x)).
\]

Surely \(f\) is a join-homomorphism maintaining \(0\) and \(1\) between \(\Phi\) and \(\Phi_1 \times \Phi_2\), whereas \(g\) is a meet-homomorphism between \(D\) and \(D_1 \times D_2\). Further, for any \(\phi \in \Phi\) and \(x \in D\),

\[
f(x(\phi)) = (f_1(x(\phi)), f_2(x(\phi)) = (g_1(x)(f_1(\phi)), g_2(x)(f_2(\phi)) = g(x)(f(\phi)).
\]

This shows that \((f, g)\) is a homomorphism from \((\Phi, D)\) into the product algebra \((\Phi_1 \times \Phi_2, D_1 \times D_2)\), and \((f_i, g_i) = (p_i, q_i) \circ (f, g)\) for \(i = 1, 2\). Therefore the product algebra is the direct product of the algebras \((\Phi_1, D_1)\) and \((\Phi_2, D_2)\). In Section 6.8 the question whether information algebras form a Cartesian closed category is examined.
Chapter 6

Compact and Continuous Algebras

6.1 Compact Information Algebras

Computers can treat only “finite” information. “Infinite” information can however often be approximated by “finite” elements. This aspect of finiteness is modeled in this section. It must be stressed that not every aspect of finiteness is captured. For example, no questions of computability and related issues will be treated. Many aspects discussed in this section, are also considered in domain theory. However, the one crucial feature not addressed in domain theory, is information extraction. Also domain theory places more emphasis on order, approximation and convergence of information and less on combination. So, although the subject is similar to domain theory, it is treated here with a different emphasis and goal. It will be seen that the subject is also closely related to algebraic and continuous lattices.

Consider an information algebra $(\Phi, D)$. In the set $\Phi$ of pieces of information we single out a subset $\Phi_f$ of elements which are considered to be finite. We shall explain in a moment in what respect these elements can be considered to be finite. In any way, combination of finite information pieces should again yield finite information. So $\Phi_f$ is assumed to be closed under combination. The neutral element $0$ is considered to be finite and belongs thus to $\Phi_f$. The same holds for the contradiction, that is $1 \in \Phi_f$.

May be one would expect that focusing of finite information yields also finite information, that is that $(\Phi_f, D)$ is a subalgebra of $(\Phi, D)$. Although this is reasonable enough and holds in many examples, we do not assume it in general.

Any information $\phi$ in $\Phi$ should be approximated with an arbitrary degree of precision by a system of finite pieces of information. The concept needed to formalize this is the concept of a directed set:

**Definition 6.1 Directed Set.** A nonempty subset $X$ of $\Phi$ is called di-
rected, if for any pair of elements $\phi, \psi \in X$ there exists an element $\chi \in X$ such that $\phi, \psi \leq \chi$.

A directed set $X$ of $\Phi$ is said to converge in $\Phi$, if the supremum of $X$ exists in $\Phi$. In this case we denote this supremum by $\vee X$.

In order to model approximation of an information $\phi$ in $\Phi$ by finite elements, we require first that all directed sets of finite pieces of information converge in $\Phi$. This means that the corresponding suprema are elements of $\Phi$ which are approximated by finite elements. But we want more: any element of $\Phi$ must be approximable in this way. This means that the finite elements must be dense in $\Phi$, that is any element of $\Phi$ is the supremum of a directed set of finite elements. That is, putting

$$A_\phi = \{\psi \in \Phi_f : \psi \leq \phi\},$$

which is a directed set, we require that

$$\phi = \vee A_\phi.$$  \hspace{1cm} (6.2)

These two requirements do not yet say much about finiteness. However, one thing they say, is that if $\phi$ is finite, then, according to equation (6.1) it belongs itself to the converging set, $\phi \in A_\phi$. This is surely an important property of finiteness. But again we want more. We may possibly approximate an element by a directed set $X$ of finite elements which is strictly contained in the set of all finite elements smaller than $\vee X$. But then, if $\phi$ is a finite element, $\phi \leq \vee X$, if $\phi$ belongs not to $X$, we require the existence of an element $\psi$ in $X$ smaller than $\phi$. This is a compactness property.

So we impose the following conditions on $\Phi$ and $\Phi_f$.

1. **Convergence**: If $X \subseteq \Phi_f$ is a directed set, then the supremum $\vee X$ over $X$ exists and belongs to $\Phi$.

2. **(Weak) Density**: If $\phi \in \Phi$, then

$$\phi = \vee \{\psi \in \Phi_f : \psi \leq \phi\}.$$ \hspace{1cm} (6.3)

3. **Compactness**: If $X \subseteq \Phi_f$ is directed, and $\phi \in \Phi_f$ such that $\phi \leq \vee X$, then there exists $\psi \in X$ such that $\phi \leq \psi$.

Whereas these requirements clearly should be satisfied for approximation, we need sometimes a stronger requirement regarding density, namely, that an information $\phi$ supported by some domain $x$ can be approximated by finite pieces of information supported by the same domain. This is expressed by the following strong version of density: For $x \in D$ and $\phi \in \Phi$, supported by $x$, $\phi = x(\phi)$, we require that

$$x(\phi) = \vee \{\psi \in \Phi_f : \psi = x(\psi) \leq \phi\}.$$ \hspace{1cm} (6.4)
Below we shall see an example of an information algebra satisfying the requirement of density, but not the strong density (Example 6.4). This shows that (6.3) does not imply (6.4). We shall see below, that a compact information as defined in the following definition, essentially makes $\Phi$ an algebraic lattice (see Theorem 6.2), supplemented with a family of extraction operators.

**Definition 6.2 Compact Information Algebra.** An information algebra $(\Phi, D)$ is called compact, if there is a subset $\Phi_f \subseteq \Phi$, which is closed under join, contains $0$ and $1$, and such that the conditions of convergence, density and compactness are satisfied for $\Phi_f$. The elements of $\Phi_f$ are called finite.

The following strengthening of this definition, adds an essential new element, and extends compact information algebra beyond domain theory.

**Definition 6.3 $D$-Compact Information Algebra.** An information algebra $(\Phi, D)$ is called $D$-compact, if it is compact and (6.4) holds for all $\phi \in \Phi$ and $x \in D$.

Here are a few examples of compact information algebras.

**Example 6.1 Finite Information Algebras:** If the information algebra $(\Phi, D)$ is finite, i.e. if $\Phi$ has only finitely many elements, then it is (trivially) compact. In fact all elements are finite, $\Phi_f = \Phi$. Convergence and density, even strong density, clearly hold. If $X$ is a (necessarily finite), directed set, $X = \{\phi_1, \ldots, \phi_m\}$, then $\phi_1 \vee \cdots \vee \phi_m \in X$, by the directedness of $X$. This implies compactness. Of course this is not a particularly interesting example of a compact algebra.

**Example 6.2 Strings.** In the algebra of strings (see Section 3.1), the finite strings $\Sigma^*$ together with $z$ are the finite elements. In this example the finite elements $(\Sigma^*, D)$ form a subalgebra of $(\Sigma^{**}, D)$ and this algebra is $D$-compact, with $D = \omega \cup \{\infty\}$.

**Example 6.3 Convex Sets:** The convex polyhedra are the finite elements of the information algebra of convex sets in $\mathbb{R}^n$ (see Section 3.4). In this example again strong density holds: Clearly, cylindrification of convex polyhedra yields convex polyhedra and convex polyhedra supported by a given domain $x$ can be used to approximate convex sets supported by the same domain. In fact, convex polyhedra form a subalgebra of the information algebra of convex sets.

**Example 6.4 Cofinite Sets:** Consider the information algebra of subsets of $\mathbb{R}^n$ (see Section 3.4). A cofinite set in $\mathbb{R}^n$ is the complement of a finite
set. Cofinite sets are closed under intersection. Any subset of \( \mathbb{R}^n \) is the intersection of all cofinite sets it is contained in. This indicates that the cofinite sets of \( \mathbb{R}^n \) are the finite elements of the information algebra of subsets of \( \mathbb{R}^n \). But strong density does not hold, since the cylindrifications of cofinite sets to subspaces of \( \mathbb{R}^n \) are no more cofinite. So, this algebra is not strongly compact.

Example 6.5  Compact Information Systems. An information system \((\mathcal{L}, \vdash)\) together with the family of sublanguages \(\mathcal{S}\) (see Section 4.2) is called compact if the consequence operator \(C\) satisfies the condition

\[(C6) \quad C(X) = \bigcup \{C(Y) : Y \subseteq X, Y \text{ finite}\}.
\]

This is equivalent to the following condition on the underlying entailment relation: If \(X \vdash s\), then there is a finite subset \(Y\) of \(X\) such that \(Y \vdash s\). We assume that the consequence operator satisfies conditions (C4) and (C5) of Section 4.2 such that the operator defines an associated information algebra \((\Phi, D)\). If in addition \(C\) satisfies also (C6), then this information algebra becomes \(D\)-compact, as Theorem 6.1 shows.

Theorem 6.1  Let \(C\) be a compact consequence operator, which also satisfies (C4) and (C5) of Section 4.2. Then the information algebra of its closed sets \((\Phi, D)\) is \(D\)-compact with finite elements

\[\Phi_f = \{C(X) : X \subseteq \mathcal{L}, X \text{ finite}\}.
\]

In fact, if \(D\) is a directed set of elements in \(\Phi_f\), then

\[\bigvee D = \bigcup D.
\]

Proof. (Kohlas, 2003a). Note that \(C(\emptyset) \in \Phi_f\). Further, if \(C(Y_1), C(Y_2) \in \Phi_f\), then clearly \(C(Y_1) \vee C(Y_2) = C(Y_1 \cup Y_2) \in \Phi_f\). It remains to verify the conditions of convergence, density, strong density and compactness.

In Section 4.2 we have seen that \(\Phi\) is a complete lattice. This implies convergence. In fact, we show more: Assume \(D\) a directed set in \(\Phi_f\) and \(s \in C(\bigcup D)\). Since \(C\) is compact, there exists a finite subset \(X \subseteq \bigcup D\) such that \(s \in C(X)\). Each element of \(X\) is in at least one element of \(D\), hence, since \(D\) is directed, there exists a set \(E \in D\) such that \(X \subseteq E\). So \(s \in C(X) \subseteq E \subseteq \bigcup D\). Therefore we conclude that \(C(\bigcup D) \subseteq \bigcup D\), thus \(C(\bigcup D) = \bigcup D\) which proves (6.5).

Note that (C6) is already the density condition. Let \(L \in \mathcal{S}\) and \(X \in \Phi\). Then by the compactness of \(C\),

\[c_L(X) = C(X \cap L) = \bigcup \{C(Y) : Y \subseteq X \cap L, Y \text{ finite}\}.
\]
Now, \( Y \subseteq L \) implies \( Y \subseteq C(Y) \cap L \), hence \( C(Y) \subseteq C(C(Y) \cap L) \subseteq C(Y) \), hence \( C(Y) = C(C(Y) \cap L) = c_L(C(Y)) \). Therefore,
\[
c_L(X) = \bigcup \{ F \in \Phi_f : F \subseteq X, F = c_L(F) \}.
\]
This is the strong density condition.

Assume that \( D \subseteq \Phi_f \) is directed, \( C(Y) \in \Phi_f \) and \( C(Y) \subseteq \cup D \). Since \( Y \subseteq C(Y) \) and \( D \) is directed, there is a set \( F \in D \) such that \( Y \subseteq F \). Hence, we see that \( C(Y) \subseteq F \). This proves the compactness condition.

As mentioned above, compact information algebras are essentially algebraic lattices, supplemented by information extraction operators. This is stated in the following theorem.

**Theorem 6.2** Let \( (\Phi, D) \) be a compact information algebra. Then the following holds:

1. \( \Phi \) is a complete lattice.
2. We have \( \phi \in \Phi_f \) if, and only if, for every directed set \( X \subseteq \Phi \), \( \phi \leq \lor X \) implies that there is a \( \psi \in X \) such that \( \phi \leq \psi \).
3. We have \( \phi \in \Phi_f \) if, and only if, for every subset \( X \subseteq \Phi \), \( \phi \leq \lor X \) implies that there is a finite subset \( F \subseteq X \) such that \( \phi \leq \lor F \).

**Proof.** (1) Let \( X \) be a non-empty subset of \( \Phi \). We show that \( X \) has an infimum. It then follows by standard results from order theory that \( \Phi \) is a complete lattice. Let \( Y \) be the set of all finite lower bounds of \( X \). It is obviously a directed set, so \( \lor Y \) exists by the convergence property. It is a lower bound of \( X \). Let \( \psi \) be another lower bound of \( X \). Then by the density condition \( \psi = \lor A_\psi \). But \( A_\psi \) is a subset of \( Y \), hence \( \psi \leq \lor Y \), hence the latter equals \( \land X \).

(2) Assume first that \( \phi \in \Phi_f \) and let \( X \) be a directed subset of \( \Phi \) such that \( \phi \leq \lor X \). Define
\[
Y = \{ \psi \in \Phi_f : \exists \chi \in X, \psi \leq \chi \}.
\]
Since \( X \) is directed, \( Y \) is directed too. If \( \chi \) is an element of \( X \), then \( A_\chi \) is a subset of \( Y \), hence \( \chi = \lor A_\chi \leq \lor Y \) and so \( \phi \leq \lor X \leq \lor Y \). By the compactness condition, there is then a \( \psi \in Y \) so that \( \phi \leq \psi \), hence a \( \chi \in X \), so that \( \phi \leq \psi \leq \chi \). This proves one direction of (2).

For the converse direction consider the directed set \( A_\phi \). Since \( \phi = \lor A_\phi \) and \( A_\phi \) is a directed set, there exists a \( \psi \in A_\phi \) such that \( \phi \leq \psi \). From the definition of \( A_\psi \) it follows that \( \phi = \psi \in \Phi_f \) and this proves (2).
The third assertion follows from (2) by the following observation: Let $X$ be an arbitrary subset of $\Phi$ and define

$$Z = \{ \lor Y : Y \subseteq X, Y \text{ finite} \}.$$ 

Now, $Z$ is not empty, since $0 = \lor \emptyset \in Z$. If $Y_1$ and $Y_2$ are both finite subsets of $X$, then so is $Y_1 \cup Y_2$ and $\lor(Y_1 \cup Y_2)$ is an upper bound of $\lor Y_1$ and $\lor Y_1$. So $Z$ is directed. Obviously $\lor Z \leq \lor X$. But $X$ is contained in $Z$, since $\phi = \lor \{ \phi \}$. So we obtain finally that $\lor Z = \lor X$.

Assume now that $\phi \in \Phi_f$ and $\phi \leq \lor X = \lor Z$. Then by (2) there is a finite subset $Y$ of $X$ so that $\phi \leq \lor Y$. Conversely, assume that $\phi \leq \lor X = \lor Z$ and that $Y$ is a finite subset of $X$ such that $\phi \leq \lor Y$. From $\lor Y \in Z$ it follows by (2) that $\phi \in \Phi_f$. \hfill \Box

Issue (2) in the theorem above corresponds to the definition of finite elements in a complete partially ordered set (cpo), whereas (3) is the definition of a compact element in a cpo. Issues (1) and (2), or (1) and (3) together mean that $\Phi$ is an algebraic lattice, where finiteness and and compactness of elements coincide (Davey & Priestley, 1990). The theorem says that our finite elements in $\Phi_f$ are exactly the finite and compact elements of $\Phi$ in the usual order-theoretic sense. Further $\Phi - \{ 1 \}$ is an algebraic complete order such that $\phi \lor \psi \in \Phi - \{ 1 \}$ if $\phi \lor \psi \neq 1$. Such a structure is also called a Scott-Ershov domain. So compact information algebras are essentially algebraic lattices supplemented with extraction operators $x \in D$.

More interesting are $D$-compact information algebras. $D$-compact information algebras turn out to be a genuinely new concept, with an interesting theory. Here is a first result, which expresses a continuity property of the operators $x \in D$ (see Section 6.7).

**Theorem 6.3** Let $(\Phi, D)$ be a $D$-compact information algebra with finite elements $\Phi_f$. If $X$ is a directed subset of $\Phi$ and $x \in D$, then

$$x(\lor X) = \lor x(\phi).$$

**Proof.** If $\phi \in X$, then $\phi \leq \lor X$, hence $x(\phi) \leq x(\lor X)$ and therefore $\lor_{\phi \in X} x(\phi) \leq x(\lor X)$.

Conversely, by the strong density (6.4),

$$x(\lor X) = \lor \{ \psi \in \Phi_f : \psi = x(\psi) \leq \lor X \}.$$ 

But $\psi \leq \lor X$ implies by (2) of Theorem 6.2 that there is a $\phi \in X$ such that $\psi \leq \phi$. Then $\psi = x(\psi) \leq x(\phi)$, hence $x(\lor X) \leq \lor_{\phi \in X} x(\phi)$, so (6.6) holds. \hfill \Box

Next, we show that strong density implies density for elements which are supported by some $x \in D$. Hence, if $(\Phi, D)$ is supported, then strong density implies weak density. The converse is not always true as an example below shows.
6.2. REPRESENTATION THEOREM

**Theorem 6.4** Let \((\Phi, D)\) be an information algebra satisfying (6.4). Then, for \(\phi \in \Phi\) and \(x \in D\) such that \(\phi = x(\phi)\), the weak density condition holds.

**Proof.** By the strong density condition (6.4)

\[
\phi = x(\phi) = \lor \{ \psi \in \Phi_f : \psi = x(\psi) \leq \phi \} \\
\leq \lor \{ \psi \in \Phi_f : \psi \leq \phi \} \leq \phi.
\]

So, \(\phi = \lor \{ \psi \in \Phi_f : \psi \leq \phi \} \) and weak density holds for \(\phi\).

In general however weak density is strictly weaker than strong density as the example of cofinite sets above shows. Here is another example of the same category:

**Example 6.6** A compact information algebra, which is not strongly compact (Kohlas, 2003a). Let \(\Phi = \{0, 1, 2, \ldots\} \cup \{\infty\}\) and \(D = \{o_0, o_1\}\) and define join as maximum, and extraction as follows:

\[
o_1(\phi) = \phi, \quad o_0(\phi) = \begin{cases} 0, & \text{if } \phi \neq \infty, \\ \infty, & \text{otherwise}. \end{cases}
\]

Take \(\{0, 1, 2, \ldots\}\) as the finite elements. Then \((\Phi, D)\) is a compact information algebra. But it is not \(D\)-compact, since \(\infty\) is supported by \(o_0\), but there are no finite elements supported by \(o_0\), except the element 0.

We note that the subalgebras \((Fix_x, D)\) of a \(D\)-compact information algebra \((\Phi, D)\) are themselves \(D\)-compact with \(\Phi_f \cap Fix_x\) as finite elements for all \(x \in D\). In fact, if \(X \subseteq Fix_x\) is directed, then by Theorem 6.3,

\[
x(\lor X) = \lor_{\phi \in X} x(\phi) = \lor_{\phi \in X} \lor X = \lor X,
\]

and therefore \(\lor X \in Fix_x\). So convergence holds. Further, for \(\phi \in Fix_x\) and any \(y \in D\),

\[
y(\phi) = y(x(\phi)) = (x \land y)(\phi) = \lor \{ \psi \in \Phi_f : \psi = (x \land y)(\psi) \leq \phi \}
\]

\[
= \lor \{ \psi \in \Phi_f \cap Fix_x : \psi = y(\psi) \leq \phi \},
\]

since \(\psi = (x \land y)(\psi)\) implies \(\psi = x(\psi)\). So, strong density holds in \(Fix_x\). Compactness in \(Fix_x\) is inherited from \(\Phi\).

6.2 Representation Theorem

Compact information algebras can be obtained from any information algebra \((\Phi, D)\) by ideal completion.
Theorem 6.5 If $(\Phi, D)$ is an information algebra, then its ideal completion $(I_\Phi, D)$ is a $D$-compact information algebra with the set $\{\downarrow \phi : \phi \in \Phi\}$ of principal ideals as finite elements.

Proof. Since the map $\phi \mapsto \downarrow \phi$ is an embedding of $(\Phi, D)$ into $(I_\Phi, D)$, we identify any element $\phi \in \Phi$ with the principal ideal $\downarrow \phi$ in $I_\Phi$. Note that $\downarrow \phi \leq I$ is then equivalent to $\phi \in I$ for any ideal $I$. It remains to prove convergence, (weak) density and compactness relative to $\Phi$ as finite elements of $I_\Phi$.

Convergence: Let $X \subseteq \Phi$ be a directed set. We claim that $I(X) = \vee X$, where $I(X)$ is the ideal generated by $X$. Then $\phi \in X$ implies $\phi \in I(X)$, hence $I(X)$ is an upper bound of $X$. Let $I$ be another upper bound of $X$. Then $X \subseteq I$, hence $I(X) \subseteq I$. Thus $I(X)$ is the least upper bound of $X$ in $I_\Phi$.

Density: Consider any ideal $I$ in $I_\Phi$ and let $X = \{\phi \in \Phi : \phi \leq I\}$, then $X = I$ and $\vee X = I$. This is (weak) density. Further, by definition, we have for an ideal $I$ of $\Phi$ and $x \in D$,

$$x(I) = \{\psi \in \Phi : \psi \leq x(\phi) \text{ for some } \phi \in I\}.$$  

We are going to show that $x(I) = \vee X = I(X)$ for $X = \{\phi \in \Phi : \phi = x(\phi) \leq I\}$. Consider $\psi \in I(X)$. Then there are elements $\phi_1, \ldots, \phi_n \in X$ such that

$$\psi \leq \phi_1 \vee \cdots \vee \phi_n = x(\phi_1) \vee \cdots \vee x(\phi_n) \quad (6.7)$$

Let $\phi = \phi_1 \vee \cdots \vee \phi_n$ then $\phi = x(\phi) \in I$. This shows that $\psi \in x(I)$ and $I(X) \subseteq x(I)$.

Conversely, assume $\psi \in x(I)$, hence $\psi \leq x(\phi)$ for some $\phi \in I$. But then $x(\phi) \in I$. Since $x(\phi) = x(x(\phi)) \leq I$, we conclude that $x(\phi) \in X$. Therefore we have $\psi \in I(X)$ and $x(I) \subseteq I(X)$. So we have proved that $x(I) = I(X)$ which is strong density.

Compactness: If $X \subseteq \Phi$ is directed and $\phi \leq \vee X$, then $\phi \leq \vee F$ for some finite subset $F$ of $X$. Since $X$ is directed, there is a $\psi \in X$ which is an upper bound of $F$, hence $\phi \leq \psi \in X$. □

Ideal completion is thus a way to compactify an information algebra $(\Phi, D)$. There is another, different way to achieve the same result. The following approach has been proposed in (Guan & Li, 2010b): Let $Di_\Phi$ denote the family of directed sets in $\Phi$ and define for $X, Y \in Di_\Phi$ and $x \in D$

$$X \otimes Y = \{\phi \vee \psi : \phi \in X, \psi \in Y\},$$

$$x((X) = \{x(\phi) : \phi \in X\}.$$  

The following lemma shows that these operations yield again directed sets.

Lemma 6.1 The sets $X \otimes Y$ and $x(X)$ belong both to $Di_\Phi$, if $X, Y \in Di_\Phi$ and $x \in D$.
6.2. REPRESENTATION THEOREM

Proof. Assume \( \eta_1 = \phi_1 \lor \psi_1 \) and \( \eta_2 = \phi_2 \lor \psi_2 \) both in \( X \otimes Y \). Then \( \phi_1, \phi_2 \in X \), hence there is a \( \phi \in X \) such that \( \phi_1, \phi_2 \leq \phi \), since \( X \) is directed. Similarly, there is a \( \psi \in Y \) such that \( \psi_1, \psi_2 \leq \psi \). Therefore we conclude that \( \eta_1, \eta_2 \leq \phi \lor \psi \in X \otimes Y \). This shows that \( X \otimes Y \) is directed.

Similarly, consider \( \phi_1, \phi_2 \in x(X) \). Then we have \( \phi_1 = x(\psi_1) \) and \( \phi_2 = x(\psi_2) \) for some elements \( \psi_1, \psi_2 \in X \). Since \( X \) is directed there is a \( \psi \in X \) such that \( \psi_1, \psi_2 \leq \psi \). But then also \( \phi_1, \phi_2 \leq x(\psi) \in x(X) \). This proves that \( x(X) \) is directed. \( \square \)

We define now \( X \equiv_\theta Y \), if and only if a) for all \( \phi \in X \) there is a \( \psi \in Y \) such that \( \phi \leq \psi \) and b) for all \( \psi \in Y \) there is a \( \phi \in X \), such that \( \psi \leq \phi \). Then, if \( X \) has a supremum in \( \Phi \) and \( X \equiv_\theta Y \) it follows that \( \lor X = \lor Y \).

In fact, by the definition of \( \equiv_\theta \), \( \lor X \) is an upper bound of \( Y \) and any other upper bound of \( Y \) must also be an upper bound of \( \lor X \). So, relation \( \equiv_\theta \) groups directed sets with the same supremum (if it exists) together. It is obviously an equivalence relation in \( Di_\Phi \). Note also that if \( X \) is any set containing \( \phi \) and all other elements of \( X \) are smaller than \( \phi \), then \( X \) is directed and \( X \equiv_\theta \{ \phi \} \).

We consider now the equivalence classes \( Di_\Phi / \theta \) and define the following operations between these classes:

1. Combination: \( [X]_\theta \otimes [Y]_\theta = [X \otimes Y]_\theta \) for \( X, Y \in Di_\Phi \),

2. Extraction: \( x([X]_\theta) = [x(X)]_\theta \) for \( X \in Di_\Phi \) and \( x \in D \).

Thee operations are well defined: Assume \( X_1 \equiv_\theta X_2 \) and \( Y_1 \equiv_\theta Y_2 \), then we claim that \( X_1 \otimes Y_1 \equiv_\theta X_2 \otimes Y_2 \). In fact assume \( \eta_1 = \phi_1 \lor \psi_1 \in X_1 \otimes Y_1 \), hence \( \phi_1 \in X_1 \) and \( \psi_1 \in Y_1 \). Then, by the definition of the equivalence, there is a \( \phi_2 \in X_2 \) and a \( \psi_2 \in Y_2 \) such that \( \phi_1 \leq \phi_2 \) and \( \psi_1 \leq \psi_2 \). Thus it follows that \( \eta_1 \leq \phi_2 \lor \psi_2 \in X_2 \otimes Y_2 \). By symmetry we see also that if \( \eta_2 \in X_2 \otimes Y_2 \), there is an element in \( X_1 \otimes Y_1 \) which dominates \( \eta_2 \).

Similarly, \( X \equiv_\theta Y \) implies \( x(X) \equiv_\theta x(Y) \). Consider \( \eta = x(\phi) \in x(X) \), hence \( \phi \in X \). Then there is a \( \psi \in Y \) such that \( \phi \leq \psi \), thus \( \eta \leq x(\psi) \in x(Y) \). By symmetry for \( \eta = x(\psi) \in x(Y) \) there is a \( \phi \in X \) such that \( \eta \leq x(\phi) \in x(X) \).

With these operations, \( (Di_\Phi / \theta, D) \) becomes a compact information algebra, in fact a compact information algebra isomorphic to the ideal completion \( (I_\Phi, D) \).

Theorem 6.6 With the operations defined above, \( (Di_\Phi / \theta, D) \) is a D-compact information algebra, isomorphic to \( (I_\Phi, D) \).

Proof. We show first that any directed set \( X \) in \( Di_\Phi \) equivalent is to the ideal \( I(X) \) in \( \Phi \) it generates \( X \equiv_\theta I(X) \): Consider a \( \phi \in X \), then \( \phi \leq \phi \in I(X) \). Conversely assume \( \psi \in I(X) \). Then there is a finite set of elements \( \psi_1, \ldots, \psi_n \) in \( X \) such that \( \psi \leq \psi_1 \lor \cdots \lor \psi_n \). Since \( X \) is directed
there is an element $\phi$ in $X$ which dominates all $\psi_i$, hence $\psi \leq \phi \in X$. Any equivalence class $[X]_\theta$ in $Di_\Phi/\theta$ may be represented by its ideal $I(X)$, that is $[X]_\theta = [I(X)]_\theta$.

Consider now the map $[X]_\theta \mapsto I(X)$. By the last remark, this map is well defined. It is onto $I_\Phi$: Any ideal $I$ in $\Phi$ is a directed set in $\Phi$ and $[I]_\theta \mapsto I(I) = I$. It is also one-to-one: Suppose that $I(X) = I(Y)$ and consider an element $\phi \in X$. Then there is a finite set of elements $\psi_1, \ldots, \psi_n$ in $Y$ such that $\phi \leq \psi_1 \lor \cdots \lor \psi_n$. Since $Y$ is directed there is an element $\psi \in Y$ such that $\psi_1, \ldots, \psi_n \leq \psi$ and therefore we have $\phi \leq \psi$. In the same way we find that for any element $\psi \in Y$ there is a $\phi \in X$ such that $\psi \leq \phi$. So we conclude that $X \cong Y$ or $[X]_\theta = [Y]_\theta$.

Further, we show that the map preserves operations, that is that $[X]_\theta \otimes [Y]_\theta \mapsto I(X) \lor I(Y)$. For this purpose, we prove that $I(X \otimes Y) = I(X) \lor I(Y)$. Consider $\phi \in I(X \otimes Y)$. Then there is a finite set of elements $\psi_1, \ldots, \psi_n$ in $X \otimes Y$ such that $\phi \leq \psi_1 \lor \cdots \lor \psi_n$. Each of the elements $\psi_i$ is equal to a join $\psi_{1,i} \lor \cdots \lor \psi_{n,i}$, where $\psi_{1,i} \in X$ and $\psi_{2,i} \in Y$. Since both $X$ and $Y$ are directed, there is a $\psi_1 \in X$ and a $\psi_2 \in Y$ such that $\psi_{1,i} \leq \psi_1$ and $\psi_{2,i} \leq \psi_2$, each time for $i = 1, \ldots, n$. It follows then that $\phi \leq \psi_1 \lor \psi_2$, hence that $\phi \in I(X) \lor I(Y)$.

Conversely, assume $\phi \in I(X) \lor I(Y)$ such that $\phi \leq \psi_1 \lor \psi_2$ for some elements $\psi_1 \in I(X)$ and $\psi_2 \in I(Y)$. This in turn means that $\psi_1 \leq \psi_{1,1} \lor \cdots \lor \psi_{1,n}$ for some elements $\psi_{1,1}, \ldots, \psi_{1,n} \in X$ and $\psi_2 \leq \psi_{2,1} \lor \cdots \lor \psi_{2,m}$ for some elements $\psi_{2,1}, \ldots, \psi_{2,m} \in Y$. Since $X$ and $Y$ are directed there exist elements $\phi_1 \in X$ and $\phi_2 \in Y$ such that $\psi_{1,i} \leq \phi_1$ for $i = 1, \ldots, n$ and $\psi_{2,i} \leq \phi_2$ for $i = 1, \ldots, m$. This shows that $\phi \leq \phi_1 \lor \phi_2 \in X \otimes Y$ and therefore $\phi \in I(X \otimes Y)$. All this proves that $I(X \otimes Y) = I(X) \lor I(Y)$ and hence $[X]_\theta \otimes [Y]_\theta = [X \otimes Y]_\theta \mapsto I(X \otimes Y) = I(X) \lor I(Y)$.

Further it is evident that the element $[\{0\}]_\theta$ maps to $I(\{0\}) = \{0\}$ and $[\{1\}]_\theta$ maps to $I(\{1\}) = \Phi$.

Finally, we show that $x([X]_\theta = [x(X)]_\theta$ maps to $x(I(X))$ by proving that $I(x(X)) = x(I(X))$. If $\phi \in I(x(X))$, then $\phi \leq \psi_1 \lor \cdots \lor \psi_n$, where $\psi_1, \ldots, \psi_n \in x(X)$. This means that $\psi_i = x(\phi_i)$, where $\phi_i \in X$. Since $X$ is directed there is a $\chi \in X$ such that $\phi_i \leq \chi$. It follows that $\phi \leq x(\phi_1 \lor \cdots \lor \phi_n) \leq x(\phi_1 \lor \cdots \lor \phi_n) \leq x(\chi)$. So we see that $\phi \in x(I(X))$.

Conversely, if $\phi \in x(I(X))$, then there exists a $\psi \in I(X)$ such that $\phi \leq x(\psi)$ and further $\psi \leq x_1 \lor \cdots \lor x_m$ for some elements $x_1, \ldots, x_m \in X$. Since $X$ is directed there is a $\chi \in X$ such that $x_1, \ldots, x_m \leq \chi$. Therefore, we find that $\phi \leq x(\psi) \leq x(\psi_1 \lor \cdots \lor \psi_m) \leq x(\chi) \in x(X)$. So we have that $\phi \in I(x(X))$ and this shows that $I(x(X)) = x(I(X))$.

This shows that indeed $(Di_\Phi/\theta, D)$ is a compact information algebra isomorphic to the ideal completion of $(\Phi, D)$. □

Note that $(\Phi, D)$ is embedded into the $D$-compact information algebra $(Di_\Phi/\theta, D)$ by the map $\phi \mapsto [\{\phi\}]_\theta$. This second, alternative, although equivalent way, besides the ideal completion, to compactify an information
6.2. REPRESENTATION THEOREM

algebra \((\Phi, D)\), consists of adjoining the missing suprema of directed sets \(X\), represented by the equivalence classes \([X]_\theta\). This point of view is sometimes helpful.

We return to ideal completion. The \(D\)-compact information algebra \((I_\Phi, D)\) is fully determined by its finite elements \(\phi\). This holds in fact for any \(D\)-compact information algebra. This is expressed in the following representation theorem for \(D\)-compact information algebras. This is similar to well-known results in domain theory (Stoltenberg-Hansen et al., 1994).

**Theorem 6.7 Representation Theorem for \(D\)-Compact Information Algebras.** If \((\Phi, D)\) is a \(D\)-compact information algebra with finite elements \(\Phi_f\) such that \((\Phi_f, D)\) is a subalgebra of \((\Phi, D)\), then \((\Phi, D)\) is isomorphic to the ideal completion of \((\Phi_f, D)\),

\[
(\Phi, D) \cong (I_{\Phi_f}, D).
\]

**Proof.** Let \(I\) be an ideal in \(\Phi_f\). Then \(I\) is directed in \(\Phi_f\) and its supremum exists in \(\Phi\). Let \(\phi = \lor I\) and assume \(\psi \in \Phi_f, \psi \leq \phi\). Then by compactness there is a \(\chi \in I\) such that \(\psi \leq \chi\), hence \(\psi \in I\). But this shows that \(I = A_\phi = \{\psi \in \Phi_f : \psi \leq \phi\}\). Any ideal in \(I_{\Phi_f}\) is of the form \(A_\phi\) for some \(\phi \in \Phi\). This shows that the map \(\phi \mapsto A_\phi\) is onto \(I_{\Phi_f}\) and is one-to-one. It remains to show that it is also a homomorphism.

We have \(A_0 = \{0\}\) which is the bottom element in the lattice \(I_{\Phi_f}\), and \(A_1 = \Phi_f\), which is the top element in \(I_{\Phi_f}\). Clearly \(A_\phi, A_\psi \subseteq A_{\phi \lor \psi}\), hence \(A_{\phi \lor \psi}\) is an upper bound of \(A_\phi\) and \(A_\psi\) in \(I_{\Phi_f}\). Let the ideal \(I\) be another upper bound of \(A_\phi\) and \(A_\psi\). Then there is a \(\chi \in \Phi\) such that \(I = A_\chi\). Then \(\phi, \psi \leq \chi\) and therefore \(\phi \lor \psi \in A_\chi\). This shows that \(A_{\phi \lor \psi} \subseteq I\), which implies that \(A_\phi \lor A_\psi = A_{\phi \lor \psi}\). So the map \(\phi \mapsto A_\phi\) is a join-homomorphism.

For any \(x \in D\), by definition

\[
x(A_\phi) = \{\psi \in \Phi_f : \exists \chi \in A_\phi, \psi \leq x(\chi)\}.
\]

So \(x(A_\phi) \subseteq A_{x(\phi)}\). Assume \(\psi \in A_{x(\phi)}\). By density, and Theorem 6.3,

\[
x(\phi) = \lor A_{x(\phi)} = \lor_{\chi \in A_\phi} x(\chi).
\]

The set \(X = \{x(\chi) : \chi \in A_\phi\}\) is directed. In fact, \(A_\phi\) is not empty and if \(x(\chi_1), x(\chi_2) \in X\), then \(\chi_1, \chi_2 \in A_\phi\), which is directed. Hence there is a \(\chi \in A_\phi\) such that \(\chi_1, \chi_2 \leq \chi\). But then \(x(\chi_1) \lor x(\chi_2) \leq x(\chi_1 \lor \chi_2) \leq x(\chi) \in X\). Here we use that \(\Phi_f\) is closed under information extraction since \((\Phi_f, D)\) is a subalgebra of \((\Phi, D)\) by assumption. Now, \(\psi \in A_{x(\phi)}\) implies \(\psi \leq x(\phi) = \lor_{\chi \in A_\phi} x(\chi)\). By compactness there is then a \(\eta \in A_\phi\) such that \(\psi \leq x(\eta)\) which means that \(\psi \in x(A_\phi)\). Therefore we have finally
$A_{x(\phi)} = x(A_{\phi})$, which shows that the map $\phi \mapsto A_{\phi}$ is an information algebra homomorphism.

Note that for any algebraic lattice $\Phi$, the map $\phi \mapsto A_{\phi}$ is a lattice isomorphism between $\Phi$ and the topped $\cap$-structure $I_{\Phi}$ (Davey & Priestley, 1990). In particular, if $X \subseteq \Phi$ is directed, then

$$\forall X \mapsto A_{\vee X} = \bigcup_{\phi \in X} A_{\phi}.$$  

But in general in a compact algebra $x(A_{\phi}) \neq A_{x(\phi)}$, so the representation theorem does not hold for a compact information algebra.

### 6.3 Labeled Compact Algebras

Labeled information algebras are important for computational purposes. Approximation with finite elements has also some importance in this context. It is therefore of interest to study labeled compact algebras. Note that the labeled version of a compact information algebra does not seem of much interest, since no approximation of $\phi \in \Phi_x$ is possible with finite elements $\psi \in \Psi_{x,f}$ in general. First we examine therefore the labeled version of a $D$-compact information algebra $(\Phi, D)$ with finite elements $\Phi_f$. As usual (see Section 4.4) we define for every $x \in D$

$$\Psi_x = \{ (\psi, x) : \psi \in \Psi, \psi = x(\psi) \}$$

and

$$\Psi = \bigcup_{x \in D} \Psi_x.$$  

Then, $(\Psi, D)$ denotes the labeled version of the information algebra $(\Phi, D)$. Note that every $\Psi_x$ is partially ordered by $(\psi, x) \leq (\phi, x)$ if $\psi \leq \phi$. We start with

**Lemma 6.2** Let $(\Phi, D)$ be a $D$-compact information algebra. For all $x \in D$ and for every subset $X \subseteq \text{Fix}_x$, we have

$$\bigvee_{\psi \in X} (\psi, x) = (\vee X, x).$$

*Proof.* Since $(\text{Fix}_x, D)$ is compact (see the end of Section 6.1), $\phi = \vee X$ belongs to $\text{Fix}_x$, so $(\vee X, x) \in \Psi_x$. Obviously, $(\phi, x)$ is an upper bound of all $(\psi, x)$ with $\psi \in X$. Let $(\chi, x)$ be another upper bound. Then $\psi \leq \chi$ for all $\psi \in X$, hence $\phi \leq \chi$ and therefore $(\phi, x)$ is the least upper bound, $\bigvee_{\psi \in X} (\psi, x) = (\phi, x).$  

Fix $x \in D$. Then we show that convergence, density and compactness hold relative to the set of elements

$$\Psi_{x,f} = \{ (\psi, x) \in \Psi_x : \psi \in \Phi_f \}. $$
For convergence, observe that the set \( \Psi_{x,f} \) is clearly closed under combination or join. If \( X' \subseteq \Psi_{x,f} \) is directed, then so is \( X = \{ \psi \in \Phi_f : (\psi, x) \in X \} \). Further \( X \) belongs to \( Fix_x \). By Lemma 6.2 \( \vee X' = \vee_{\psi \in X'} (\psi, x) = (\vee X, x) \in \Psi_x \). So convergence holds in \( \Psi_x \).

For density, let \( (\phi, x) \in \Psi_x \). Then, again by Lemma 6.2
\[
\vee \{ (\psi, x) \in \Psi_{x,f} : (\psi, x) \leq (\phi, x) \} = (\vee \{ \psi \in \Phi_f : \psi = x(\psi) \leq \phi \}, x) = (\phi, x)
\]
This is the density condition in \( \Psi_x \).

Finally, for compactness, assume \( X' \subseteq \Psi_{x,f} \) directed and \( (\psi, x) \in \Psi_{x,f} \), \( (\psi, x) \leq \vee X' \). Define again \( X = \{ \psi \in \Phi_f : (\psi, x) \in X \} \). Then, by Lemma 6.2 \( \phi \leq \vee X \). Therefore there is a \( \psi \in X \) such that \( \phi \leq \psi \). Then \( (\psi, x) \in X' \) and \( (\phi, x) \leq (\psi, x) \). This is compactness.

Let
\[
\Psi_f = \bigcup_{x \in D} \Psi_{x,f}.
\]
Clearly \( \Psi_f \) is closed under combination. In fact, if \( (\Phi_f, D) \) is a subalgebra of \( (\Phi, D) \), then \( (\Psi_f, D) \) is a subalgebra of the labeled algebra \( (\Psi, D) \), since \( (\phi, x) \otimes (\psi, y) = (\phi \lor \psi, x \lor y) \) and \( (\phi, x) \downarrow y = (y(\phi), y) \) (see Section 4.4) and \( \phi, \psi \in \Phi_f \) implies \( \phi \lor \psi, y(\phi) in \Phi_f \) if \( (\Phi_f, D) \) is a subalgebra of \( (\Phi, D) \).

This motivates the following definition of a labeled compact algebra:

**Definition 6.4 Labeled Compact Information Algebra:** A labeled information algebra \( (\Psi, D) \) is called compact with finite elements
\[
\Psi_f = \bigcup_{x \in D} \Psi_{x,f} \subseteq \Psi
\]
if \( \Psi_f \) is closed under combination, and for all \( x \in D \) the following conditions are satisfied:

1. **Convergence:** For all \( X \subseteq \Psi_{x,f} \) directed, \( \vee X \in \Psi_x \).
2. **Density:** For all \( \phi \in \Psi_x \), \( \phi = \vee \{ \psi \in \Psi_{x,f} : \psi \leq \phi \} \).
3. **Compactness:** If \( X \subseteq \Psi_{x,f} \) directed, then if \( \phi \in \Psi_{x,f} \), and \( \phi \leq \vee X \), there is a \( \psi \in X \) such that \( \phi \leq \psi \).

So, to a \( D \)-compact information algebra, a labeled compact information algebra is associated.

Here follow two simple examples:

**Example 6.7 Strings:** We refer to Section 3.1 and Example 6.2. The information algebra of strings is \( \omega \cup \{ \infty \} \)-compact with the finite strings as finite elements. So, its labeled version is compact. Note that for a domain \( n \in \omega \) the set \( \Psi_n \) contains all pairs \( (s, n) \) with \( |s| \leq n \) and \( \Psi_n = \Psi_{n,f} \), all elements of the domain are finite. For the domain \( \infty \), only the pairs \( (s, n) \) with finite string \( s \) are finite.
Example 6.8 Convex Sets: According to Example 6.3 convex sets in $\mathbb{R}^n$ form a compact information algebra with convex polyhedra as finite elements. Its labeled version corresponds to convex sets on $\mathbb{R}^s$ for subsets $s$ of $\{1, 2, \ldots, n\}$ and the finite elements are still the convex polyhedra of these spaces, see also Example 4.3 for more details.

A further class of examples of labeled compact information algebras is presented in Section 6.6.

From the definition of labeled compact information algebras it is obvious that some results from compact information algebras carry directly over to labeled algebras. One of these result is the one regarding the relation to algebraic lattices (see Theorem 6.2). So, it is clear that any $\Psi_x$ is an algebraic lattice, and that the finite elements $\Psi_{x,f}$ are also finite and compact in the sense of ordinary order theory. Further, Theorem 6.5 on ideal completion holds also for labeled information algebras: If $(\Psi, D)$ is a labeled information algebra and $I_{\Psi_x}$ denotes the set of ideals in the semilattice $\Psi_x$, then $(I_{\Psi_x}, D)$ with

$$I_{\Psi_x} = \bigcup_{x \in D} I_{\Psi_x}$$

is a labeled compact information algebra. The Representation Theorem 6.7 holds also in a labeled version: If $(\Psi, D)$ is a labeled compact information algebra such that its finite elements form a subalgebra $(\Psi_f, D)$, then the algebra $(\Psi, D)$ is isomorph to the ideal completion of its finite elements:

$$(\Psi, D) \cong (I_{\Psi_f}, D).$$

Here is another useful result, which is similar to Theorem 6.6:

Theorem 6.8 Let $(\Psi, D)$ be a labeled compact information algebra, $X \subseteq \Psi_y$ directed and $x \leq y$. Then

$$(\forall X)^{lx} = \forall X^{lx},$$

where $X^{lx} = \{ \psi^{lx} : \psi \in X \}$.

Proof. Assume first $\phi \in X$. Then $\phi \leq \forall X$, hence $\phi^{lx} \leq (\forall X)^{lx}$. This shows that $\forall X^{lx} \leq (\forall X)^{lx}$.

Conversely, by density in $\Psi_x$,

$$(\forall X)^{lx} = \forall \{ \psi \in \Psi_{x,f} : \psi \leq (\forall X)^{lx} \} = \forall \{ \psi \in \Psi_{x,f} : \psi^{ly} \leq \forall X \}.$$

By compactness in $\Psi_y$ there is for all $\psi^{ly} \leq \forall X$ a $\phi \in X$ such that $\psi^{ly} \leq \phi$. But then it follows that $\psi = (\psi^{ly})^{lx} \leq \phi^{lx} \in X^{lx}$, hence we see that $(\forall X)^{lx} \leq \forall X^{lx}$, hence $(\forall X)^{lx} = \forall X^{lx}$. \[\square\]

From a $D$-compact information algebra $(\Phi, D)$ we have derived a labeled compact information algebra $(\Psi, D)$. It is conceivable, that vice versa
a labeled compact information algebra \((\Psi, D)\) generates a \(D\)-compact information algebra \((\Psi/\sigma, D)\) and that \(\Psi_f/\sigma\) are its finite elements. This is in fact true under the restricting condition that \(D\) has a top element \(\top\). Note that by \(\Psi_f/\sigma\) we denote the equivalence classes of the equivalence relation \(\equiv_\sigma\) in \(\Psi\) (not only in \(\Psi_f\)). However, if \((\Psi_f, D)\) is a subalgebra of \((\Psi, D)\), then all elements of the equivalence class \([\psi]_\sigma\) belongs to \(\Psi_f\) if \(\psi \in \Psi_f\). We remark also, that if \(\psi \in \Psi_{x,f}\), then \(\psi^\top \in \Psi_{\top,f}\), a vacuous extension of a finite element remains finite, and if \((\Psi_f, D)\) is a subalgebra of \((\Psi, D)\), then also \(\psi^{\top x} \in \Psi_f\) for all \(x \in D\).

**Theorem 6.9** If \((\Psi, D)\) is a labeled compact information algebra with finite elements \(\Psi_f\) such that \((\Psi_f, D)\) is a subalgebra of \((\Psi, D)\), and \(D\) has a top element \(\top\), then \((\Psi/\sigma, D)\) is \(D\)-compact with \(\Psi_f/\sigma\) as the set of finite elements. Further \((\Psi_f/\sigma, D)\) is a subalgebra of \((\Psi/\sigma, D)\).

**Proof.** Since \(\Psi_f\) is closed under combination, \(\Psi_f/\sigma\) is closed under join: consider \([\phi]_\sigma\) and \([\psi]_\sigma\) with \(\phi, \psi \in \Psi_f\). Then \([\phi]_\sigma \vee [\psi]_\sigma = [\phi \otimes \psi]_\sigma\). But \(\phi \otimes \psi\) belongs to \(\Psi_f\), hence \([\phi \otimes \psi]_\sigma\) belongs to \(\Psi_f/\sigma\).

In order to prove convergence, strong density and compactness we need the following lemma:

**Lemma 6.3** Let \((\Psi, D)\) be a labeled information algebra. Then, for all subsets \(X\) of \(\Psi\) for which \(\forall X\) exists,

\[
[\forall X]_\sigma = \forall [X]_\sigma, \quad \text{where } \forall [X]_\sigma = \forall \{[\phi]_\sigma : \phi \in X\}.
\]

(6.9)

**Proof.** Define \(\phi = \forall X \in \Psi\), so that \([\phi]_\sigma = [\forall X]_\sigma\). Let \(d(\phi) = x\). Then for all \(\psi \in X\) from \(\psi \leq \phi\) it follows that \(d(\psi) \leq \phi\). Define therefore

\[
X^{\top x} = \{\psi^{\top x} : \psi \in X\}.
\]

(6.10)

Now, for \(\psi \in X\), we have \(\psi \leq \psi^{\top x} \leq \phi^{\top x} = \phi\) and therefore, \(\phi = \forall X \leq \forall X^{\top x} \leq \phi^{\top x} = \phi\). Hence we have \(\forall X^{\top x} = \forall X = \phi\). Further, if \(\psi \in X\), then \([\psi]_\sigma \leq [\phi]_\sigma\), thus \(\forall [X]_\sigma \leq [\phi]_\sigma\). Assume \([\chi]_\sigma\) to be another upper bound of \([X]_\sigma\). Let \(d(\chi) = y\). For a \(\psi \in X\) consider \([\psi]_\sigma \otimes [\chi]_\sigma = [\psi \otimes \chi]_\sigma = [\psi^{\top x y} \otimes \chi^{\top x y}]_\sigma = [\chi]_\sigma\). So \(\psi^{\top x y} \leq \chi^{\top x y}\) and therefore, \(\forall X^{\top x y} = \phi^{\top x y} \leq \chi^{\top x y}\). This implies \([\phi]_\sigma \leq [\chi]_\sigma\), hence \(\forall [X]_\sigma = [\phi]_\sigma\). \(\Box\)

The proof of the theorem will be continued by proving convergence. Assume \(X \subseteq \Psi_{\top, f}\) directed. For all \(\psi \in \Psi\), we have \(\psi^{\top} \in [\psi]_\sigma\) for all \(\phi \in \Psi\). Also \(\phi^{\top} \in \Psi_f\) if \(\phi \in \Psi_f\), since \(\Psi_f\) is closed under combination. Define

\[
X' = \{\phi \in \Psi_{\top, f} : [\phi]_\sigma \in X\}.
\]

(6.11)
CHAPTER 6. COMPACT AND CONTINUOUS ALGEBRAS

The set $X'$ is directed in $\Psi_{\top,f}$, hence $\forall X' \in \Psi_{\top}$. By Lemma 6.3 above we conclude that $\forall X = \forall [X']_\sigma = [\vee X']_\sigma \in \Psi / \sigma$. This proves convergence.

Next we turn to strong density. Consider $[\phi]_\sigma \in \Psi / \sigma$ and $x \in D$. Then $x([\phi]_\sigma) = [\phi^{-x}]_\sigma$. By the convergence property in $\Psi_x$, 

$$\phi^{-x} = \vee \{ \psi \in \Psi_{x,f}; \psi \leq \phi^{-x} \}. \quad (6.12)$$

Then, Lemma 6.3 shows that

$$x([\phi]_\sigma) = [\vee \{ \psi \in \Psi_{x,f}; \psi \leq \phi^{-x} \}]_\sigma = \vee \{ [\psi]_\sigma : \psi \in \Psi_{x,f}, \psi \leq \phi^{-x} \}. \quad (6.13)$$

But $\psi \in \Psi_{x,f}, \psi \leq \phi^{-x}$ if and only if $[\psi]_\sigma \in \Psi_{f / \sigma}$, $[\psi]_\sigma = x([\psi]_\sigma) \leq x([\phi]_\sigma)$. From this we conclude that

$$x([\phi]_\sigma) = \vee \{ [\psi]_\sigma \in \Psi_{f / \sigma}; [\psi]_\sigma = x([\psi]_\sigma) \leq [\phi]_\sigma \}. \quad (6.14)$$

This is the strong density condition in $\Psi / \sigma$.

Finally, in order to prove compactness, let $X \subseteq \Psi_{f / \sigma}$ be directed, $[\phi]_\sigma \in \Psi_{f / \sigma}$ and $[\phi]_\sigma \leq \vee X$. Define $X'$ as above as the set of representants of elements of $X$ on domain $\top$. We may also consider a representant $\phi$ of $[\phi]_\sigma$ with $d(\phi) = \top$. Since $X'$ is still directed, $\phi \in \Psi_{\top,f}$ and $\phi \leq \vee X'$, it follows from the compactness in $\Psi_{\top}$ that there is an element $\psi \in X'$ so that $\phi \leq \psi$. This implies $[\psi]_\sigma \in X$ and $[\phi]_\sigma \leq [\psi]_\sigma$ and this proves the compactness condition in $\Psi / \sigma$.

It remains to verify that $(\Psi_{f / \sigma}, D)$ is a subalgebra of $(\Psi / \sigma, D)$ if $(\Psi_f, D)$ is a subalgebra of $(\Psi, D)$. We have already shown that $\Psi_{f / \sigma}$ is closed under join. Clearly $[0_x]_\sigma, [1_x]_\sigma$ belong to $\Psi_{f / \sigma}$. Further, if $x \in D$ and $[\psi]_\sigma \in \Psi_{f / \sigma}$, then $x([\psi]_\sigma) = [\psi^{-x}]_\sigma$ and $\psi^{-x} \in \Psi_{x,f}$ since $(\Psi_f, D)$ is a subalgebra. But this implies also that $x([\psi]_\sigma) \in \Psi_{f / \sigma}$. So $\Psi_{f / \sigma}$ is also closed under extraction.

So, in summary, any $D$-compact information algebra $(\Phi, D)$ generates a labeled compact information algebra $(\Psi, D)$. If the finite elements of $(\Phi, D)$ form a subalgebra, then the finite elements of $(\Psi, D)$ form then also a subalgebra. Conversely, a compact labeled information algebra $(\Psi, D)$, whose finite elements form a subalgebra generate a compact information algebra $(\Psi / \sigma, D)$, whose finite elements form also a subalgebra. This is true if $D$ has a top element. But it is not true, if $D$ does not have a top element. This is shown in the following example.

Example 6.9 Convex Sets: We reconsider the example of convex sets over finite sets of variables examined in Example 4.3. We have seen in
this example that these convex sets form a labeled information algebra. In Example 6.8 we have seen that this labeled algebra is compact with convex polyhedra as finite elements. Convex polyhedra form indeed a subalgebra of the algebra of convex sets. However the lattice $D$ of finite subsets of $\omega = \{0, 1, 2, \ldots\}$ has no top element. It has been argued in Example 4.3 that the associated information algebra essentially can be seen as the algebra of cylindric subsets in $\mathbb{R}^\omega$ with base $s \in D$. The finite elements should then be the cylindric sets whose base is a convex polyhedra. However, directed sets of such cylindric polyhedra need in $\mathbb{R}^\omega$ not have a supremum which is itself a cylindric set with finite base $s$. Thus convergence does not hold and the associated information algebra is not compact.

This example shows that the top element of $D$ plays an essential role, which we want to examine a bit closer. First we note that although convergence fails in the information algebra associated with a labeled, compact algebra, $D$-density and a form of compactness remain valid.

**Theorem 6.10** If $(\Psi, D)$ is a labeled compact information algebra and $(\Psi_f, D)$ a subalgebra of it, then in $(\Psi/\sigma, D)$ $D$-density holds relative to $\Psi_f/\sigma$ and compactness holds in the sense that if $X \subseteq \Psi_f/\sigma$ is a directed set such that $\bigvee X \in \Psi_f/\sigma$ exists, then $[\psi]_\sigma \in \Psi_f/\sigma$ and $[\psi]_\sigma \leq \bigvee X$ imply that there is a $[\phi]_\sigma \in X$ such that $[\psi]_\sigma \leq [\phi]_\sigma$.

**Proof.** This follows from the proof of strong density and compactness of Theorem 6.9, since there the existence of a top element in $D$ is not used. □

So, the missing top element in $D$ makes that $(\Psi/\sigma, D)$ is not compact, even so $(\Psi, D)$ is. In Section 4.4 we have seen that we may always add a top element $\top$ to the lattice $D$ and a corresponding domain $\Psi_\top$ to $\Psi$, such that $(\Psi \cup \Psi_\top, D \cup \{\top\})$ is still a labeled information algebra. However, even if $(\Psi, D)$ is a labeled, compact information algebra, this is no more true for $(\Psi \cup \Psi_\top, D \cup \{\top\})$. The reason is, that for the same reason as in Example 6.9, convergence of directed sets of finite elements in $\Psi_\top$ does not hold.

Nevertheless, we may compactify the information algebra $(\Psi \cup \Psi_\top, D \cup \{\top\})$ in a similar way as an ordinary information algebra (see Section 6.2) by adding the missing suprema of directed sets. The approach is similar to the alternative to ideal completion described in Section 6.2 for information algebras. Let then $(\Psi, D)$ be a labeled compact information algebra with finite elements $(\Psi_f$ and suppose that $(\Psi_f, D)$ is a subalgebra of $(\Psi, D)$. Then the equivalence classes $[\psi]_\sigma$ for $\psi \in \Psi_f$ in $\Psi$ contain only elements of $\Psi_f$ as we noted above. Therefore, let $\Psi_f/\sigma$ denote the equivalence classes in $\Psi$ for $\psi \in /\Psi_f$. We assume further that $D$ has no top element. Then we adjoin a top element $\top$ to $D$ such that $D \cup \{\top\}$ remains a lattice with $x \land \top = x$ and $x \lor \top = \top$ for all $x \in D$. 

6.3. **LABELED COMPACT ALGEBRAS**

65
As a first step we add elements on a new domain corresponding to \( \top \) by adjoining an element \( \psi_\top \) for every class \([\psi]_\sigma\) for \( \psi \in \Psi_f\):

\[
\Psi_{\top,f} = \{ \psi_\top : [\psi]_\sigma \in \Psi_f/\sigma \}
\]

and define labeling, combination and projection in \((\Psi_f \cup \Psi_{\top,f}, D \cup \{\top\})\) as in Section 4.4. So, from Theorem 4.3 we conclude that \((\Psi_f \cup \Psi_{\top,f}, D \cup \{\top\})\) is a labeled information algebra. In particular, \(\Psi_{\top,f}\) is a semilattice with \(0_\top\) and \(1_\top\) as bottom and top elements.

In order to add the suprema of directed sets \(\Psi_{\top,f}\) we consider the family of directed set \(D_{i_f}\) on \(\Psi_{\top,f}\) and define \(X \equiv_\theta Y\) for \(X, Y \in D_{i_f}\) if and only if a) for all \(\phi_\top \in X\) there is a \(\psi_\top \in Y\) such that \(\phi_\top \leq \psi_\top\) and b) for all \(\psi_\top \in Y\) there is a \(\phi_\top \in X\) such that \(\psi_\top \leq \phi_\top\) (see Section 6.2). This is an equivalence relation \(\Psi_{\top,f}\). Similar as in Section 6.2 define

\[
X \otimes Y = \{ \phi_\top \otimes \psi_\top : \phi_\top \in X, \psi_\top \in Y\},
\]

\[
X^{lx} = \{ \psi_{x}^{lx} : \psi_\top \in X\} = \{ \psi^{-x} : \psi_\top \in X\}.
\]

As in the similar case of an information algebra (Section 6.2) we show that with these operation we get again directed sets.

**Lemma 6.4** The set \(X \otimes Y\) and \(X^{lx}\) are both directed, the first in \(\Psi_{\top,f}\), the latter in \(\Psi_x,f\).

**Proof.** That \(X \otimes Y \in D_{i_f}\) is shown as in Lemma 6.1. 

As for the set \(X^{lx}\) consider two elements \(\psi_{x}^{lx}_1, \psi_{x}^{lx}_2 \in X^{lx}\). Since \(\psi_{x}^{lx}_1, \psi_{x}^{lx}_2 \in X\), and \(X\) is directed, there is a \(\psi_\top \in X\) such that \(\psi_{x}^{lx}_1, \psi_{x}^{lx}_2 \leq \psi_\top\). By elementary properties of a labeled information \(\psi_{x}^{lx}_1, \psi_{x}^{lx}_2 \leq \psi_{x}^{lx}_1 \in X^{lx}\). So \(X^{lx}\) is directed.

Again, similar to the related approach for information algebras, we define operations in \(\Psi \cup D_{i_f}/\theta\) as follows:

1. **Labeling:** \(d([X]_\theta) = \top\),

2. **Combination:** \([X]_\theta \otimes [Y]_\theta = [X \otimes Y]_\theta\) and \(\psi \otimes [X]_\theta = [\{\psi_\top\}]_\theta \otimes [X]_\theta\) for \(\psi \in \Psi\),

3. **Projection:** \([X]_\theta^{lx} = \vee X^{lx}\) for \(x \in D \cup \{\top\}\).

Note that any directed set \(X\) of \(\Psi_{\top,f}\) with supremum \(\psi_\top\) is equivalent to the one element set \(\{\psi_\top\}\), \(X \equiv_\theta \{\psi_\top\}\). As in the case of information algebras (Section 6.2), these operations are well defined. For labeling there is no problem anyway. The well-definedness of combination is shown as in the case of information algebras (Section 6.2). For projection assume that \(X \equiv_\theta Y\) and a) that \(\phi^{-x} \in X^{lx}\), such that \(\phi_\top \in X\). Since \(X \equiv_\theta Y\) thee exists then a \(\psi_\top \in Y\) such that \(\phi_\top \leq \psi_\top\), hence \(\phi^{-x} \leq \psi^{-x} \in Y^{lx}\). In the
same way we show b) if \( \psi^{\rightarrow x} \in Y^{\downarrow x} \), then there is a \( \phi^{\rightarrow x} \in X^{\downarrow x} \) such that 
\( \psi^{\rightarrow x} \leq \phi^{\rightarrow x} \in X^{\downarrow x} \), hence we have \( X^{\downarrow x} \equiv_\theta Y^{\downarrow x} \). Thus, projection is well defined.

Next, we show that \((\Psi \cup Di_f/\theta, D \cup \{\top\})\) form a labeled information algebra and that the labeled information algebra \((\Psi \cup \Psi_\top, D \cup \{\top\})\) (see Section 4.4) is embedded in it.

**Theorem 6.11** If \((\Psi, D)\) is a labeled compact information algebra with finite elements \(\Psi_f\), and if \((\Psi_f, D)\) is a subalgebra of \((\Psi, D)\), then \((\Psi \cup Di_f/\theta, D \cup \{\top\})\) is a labeled information algebra and \((\Psi \cup \Psi_\top, D \cup \{\top\})\) is embedded in it.

**Proof.** We already noted that \(D \cup \{\top\}\) is a lattice. We need to verify the axioms of a labeled information algebra only insofar as elements of \(Di_f/\theta\) are involved.

Clearly combination is commutative. It is also associative. We verify this in one of the cases to be considered:

\[
(\phi \otimes \psi) \otimes [X]_\theta = ([\phi \otimes \psi]_\top \otimes [X]_\theta) = [((\phi_\top \otimes \psi_\top) \otimes \eta_\top : \eta_\top \in X)]_\theta \\
= ([\phi_\top \otimes (\psi_\top \otimes \eta_\top) : \eta_\top \in X])_\theta = ([\phi_\top] \otimes ([\psi_\top] \otimes \eta_\top : \eta_\top \in X])_\theta \\
= ([\phi_\top] \otimes ([\psi_\top] \otimes [X]_\theta))_\theta = [\phi \otimes (\psi \otimes [X]_\theta)]_\theta.
\]

The other cases are treated similarly. Further, \([0 \top]_\top\) and \([1 \top]_\top\) are obviously the bottom and top elements in \(D_f/\theta\). So axiom 2 holds (see Section 4.4).

From the definition of combination and labeling we obtain \(d(\phi \otimes [X]_\theta) = d([\phi_\top] \otimes X]_\theta) = \top\). But this equals also \(d(\phi) \lor d([X]_\theta) = \top\). So axiom 3 is also valid. Similarly we have \(d([X]_\theta^{\downarrow x}) = d(\lor X^{\downarrow x}) = x\). This is axiom 4.

If \(x \leq y \leq \top\), then, be Theorem 6.6, \((\bot_\top)\) \(\lor X^{\downarrow x} = ((\lor X^{\downarrow x})^{\downarrow x} = (\lor X^{\downarrow x}^{\downarrow x} = (\lor X)^{\downarrow x} = X^{\downarrow x} = [X]_\theta^{\downarrow x}\). This proves axiom 5.

Assume now \(d(\phi) = x\). Then again using Theorem 6.6,

\[
(\phi \otimes [X]_\theta)^{\downarrow x} = [\phi_\top \otimes [X]_\theta]^{\downarrow x} = [\phi_\top \otimes \psi_\top : \psi_\top \in X]_\theta^{\downarrow x} \\
= \lor\{(\phi_\top \otimes \psi_\top)^{\downarrow x} : \psi_\top \in X\} = \lor\{(\phi \otimes \psi)^{\rightarrow x} : \psi_\top \in X\} \\
= \lor\{(\phi \otimes \psi)^{\rightarrow x} : \psi_\top \in X\} = \phi \lor \lor\{\psi^{\rightarrow x} : \psi_\top \in X\} = \phi \lor [X]_\theta^{\downarrow x}.
\]

This proves axiom 6. Further,

\[
[X]_\theta \otimes [X]_\theta^{\downarrow x} = [X \otimes \lor X^{\downarrow x}]_\theta = [\phi_\top \otimes \psi^{\rightarrow x} : \phi_\top, \psi_\top \in X]_\theta.
\]

We claim that \(\{\phi_\top \otimes \psi^{\rightarrow x} : \phi_\top, \psi_\top \in X\} \equiv_\theta X\). In fact, a) \(\phi_\top \otimes \psi^{\rightarrow x} \leq \phi_\top \otimes \psi_\top \leq \chi_\top\) for some \(\chi_\top \in X\), since \(X\) is directed. Conversely, b) if \(\chi_\top \in X\), then \(\chi_\top \leq \chi_\top \otimes \chi^{\downarrow x}\) which belongs to the set on the left hand side of the equivalence above. Therefore

\[
[X]_\theta = [\phi_\top \otimes \psi^{\rightarrow x} : \phi_\top, \psi_\top \in X]_\theta.
\]
This verifies axiom 7. 

Finally, $[[\{0_x\}]_\theta]^{\top} = \lor [0_x] = 0_x$ and $0_x \otimes [\{1_\top\}]_\theta = [[0_\top] \otimes \{1_\top\}]_\theta = [[\{1_\top\}]_\theta$. So axioms 8 and 9 hold too.

Note that $(\psi, D)$ is in this setting a subalgebra of $(\Psi \cup Di_\top \theta, D \cup \{1\})$, is included in it. So, the identity map of $\Psi$ into $X$, together with the map $\psi \mapsto [[\psi_\top]]_\theta$ from $\Psi$ into $Di_\top / \theta$ defines an embedding of $(\Psi \cup \Psi_\top, D \cup \{1\})$ into $(\Psi \cup Di_\top / \theta, D \cup \{1\})$ as can readily be verified. This concludes the proof.

Now, $(\Psi \cup Di_\top / \theta, D \cup \{1\})$ is not only a labeled compact information algebra, but a compact one.

**Theorem 6.12** If $(\Psi, D)$ is a labeled compact information algebra with finite elements $\Psi_\top$, and if $(\Psi_\top, D)$ is a subalgebra of $(\Psi, D)$, then $(\Psi \cup Di_\top / \theta, D \cup \{1\})$ is a labeled compact information algebra with $\Psi_\top \cup \{[[\psi_\top]]_\theta : \psi \in \Psi_\top\}$ as finite elements.

**Proof.** We need only to verify the compactness condition on the added top-domain $Di_\top / \theta$. To do that we note that we may identify the elements $[[\psi_\top]]_\theta$ with $\psi_\top$ due to the embedding defined at the end of the previous proof, and identify the finite elements in $Di_\top / \theta$ by $\Psi_\top \, f$.

**Convergence:** Let $X \in \Psi_\top$ be directed. For all $\psi_\top \in X$, we have $\psi_\top \leq [X]_\theta$. So $[X]_\theta$ is an upper bound of $X$. Let $[Y]_\theta$ be another upper bound of $X$. Then $\psi_\top \in Y$ for all $\psi_\top \in X$, hence $[X]_\theta \leq [Y]_\theta$. This shows that $\lor X = [X]_\theta$, hence convergence holds.

**Density:** By convergence we have

$$\lor \{\psi_\top \in \Psi_\top, f : \psi_\top \leq [X]_\theta\} = \{\psi_\top \in \Psi_\top, f : \psi_\top \leq [X]_\theta\}_\theta \leq [X]_\theta.$$

But we have also $X \subseteq \{\psi_\top \in \Psi_\top, f : \psi_\top \leq [X]_\theta\}$, hence $\lor X = [X]_\theta$, therefore we conclude that

$$[X]_\theta = \lor \{\psi_\top \in \Psi_\top, f : \psi_\top \leq [X]_\theta\}.$$

This is the density condition.

**Compactness:** Assume $X \Psi_\top$ directed and $\psi_\top \in \Psi_\top$ such that $\psi_\top \leq \lor X \equiv [X]_\theta$. This means that $[X]_\theta = \psi_\top \otimes [X]_\theta = [[\psi_{\top \otimes \phi_\top} : \phi_\top \in X]]_\theta$. In other words, $X \equiv [\psi_{\top \otimes \phi_\top} : \phi_\top \in X]$. There is therefore a $\chi_\top \in X$ such that $\psi_\top \leq \psi_{\top \otimes \phi_\top} \leq \chi_\top$. This shows compactness.

In this way, any labeled compact information algebra $(\Psi, D)$, where $D$ has no top element, may be extended to a labeled compact information algebra $(\Psi \cup Di_\top / \theta, D \cup \{1\})$. According to Theorem 6.9 the associated information algebra $((\Psi \cup Di_\top / \theta) / \sigma, D \cup \{1\})$ is also compact. On the other hand we can extend the information algebra $((\Psi \cup \Psi_\top) / \sigma, D)$ to a compact information algebra $((\Psi \cup \Psi_\top) / \sigma, D \cup \{\theta\})$ which is isomorphic to the ideal completion of $(\Psi \cup \Psi_\top) / \sigma, D$ (Section 6.2). Both algebras contain the information algebra $((\Psi \cup \Psi_\top) / \sigma, D)$ as an embedding. We conclude that the two extended information algebras must be isomorphic.
Theorem 6.13 Let $(\Psi, D)$ be a labeled compact information algebra, where $D$ has no top element. Then, the information algebra $((\Psi \cup Di_f / \theta) / \sigma, D \cup \{\top\})$ associated with its compact extension and the information algebra $(((\Psi \cup \Psi_\top) / \sigma) / \theta, D \cup \{\theta\})$ are isomorphic.

Proof. Note first that information algebra $((\Psi \cup Di_f / \theta) / \sigma, D \cup \{\top\})$ is isomorphic to the top domain adjoined to $\Psi$ and further let $R = \{(\Psi \cup Di_f / \theta) / \sigma, D \cup \{\theta\}\}$ be the set of all suprema of directed sets in $\Psi_\top$ in $Di_f / \theta$, which have no suprema in $\Psi_\top$. Let then

$$\Psi_\top = \Psi_\top \cup R.$$ 

We design then the top algebra associated with the labeled algebra $(\Psi \cup Di_f, D \cup \{\top\})$ by $(\Psi_\top, D)$, where $\bar{D} = \{\bar{x} : x \in D \cup \{\top\}\}$ (see Theorem 4.4). We show that this algebra is isomorphic to $((\Psi \cup \Psi_\top) / \sigma) / \theta, D \cup \{\theta\})\] and further let $\mathcal{R} = \{(\Psi \cup Di_f / \theta, \bar{D} \cup \{\top\}\}$ be the set of all suprema of directed sets in $\Psi_\top$ in $Di_f / \theta$, which have no suprema in $\Psi_\top$. Let then

$$\Psi_\top = \Psi_\top \cup \mathcal{R}.$$ 

For this purpose remind that the information algebra $((\Psi \cup \Psi_\top) / \sigma, D \cup \{\top\})$ is embedded in $((\Psi \cup \Psi_\top) / \sigma) / \theta, D \cup \{\theta\})$ and as usual we identify the image of the elements of $((\Psi \cup \Psi_\top) / \sigma, D \cup \{\top\})$ with elements themselves; that is we consider that $|\psi|_\sigma \in ((\Psi \cup \Psi_\top) / \sigma) / \theta$. Further let

$$\mathcal{R}' = \{(X')_\sigma : X' \in (\Psi \cup \Psi_\top) / \sigma, \text{ directed } \vee X \notin (\Psi \cup \Psi_\top) / \sigma\},$$

So, $\mathcal{R}'$ contains the suprema of all directed sets in $(\Psi \cup \Psi_\top) / \sigma$, which do not belong to $(\Psi \cup \Psi_\top) / \sigma$. This is similar to the set $\mathcal{R}$ above. And similarly to the decomposition of $\Psi_\top$ above we have also

$$(\Psi \cup \Psi_\top) / \theta = (\Psi \cup \Psi_\top) / \sigma \cup \mathcal{R}'$$

We define now a map between $(\Psi_\top, D)$ and $((\Psi \cup \Psi_\top) / \sigma) / \theta, D \cup \{\theta\})$ by

$$\begin{align*}
\bar{x} \in \bar{D} & \mapsto x \in D \cup \{\top\}, \\
\psi_\top \in \Psi_\top & \mapsto [\psi_\top]_\sigma \in (\Psi \cup \Psi_\top) / \sigma, \\
[X]_\sigma \in \mathcal{R} & \mapsto [X']_\sigma \in \mathcal{R}', \\
\text{where } X' &= \{[\psi_\top]_\sigma : \psi_\top \in X\}.
\end{align*}$$

Clearly, the map $\bar{x} \mapsto x$ is a semilattice isomorphism between $\bar{D}$ and $D \cup \{\top\}$. The second part of the map between $\Psi_\top$ and $(\Psi \cup \Psi_\top) / \sigma / \theta$ is obviously onto and one-to-one. The first part $\psi_\top \mapsto [\psi_\top]_\sigma$ is a homomorphism by Theorem 4.4). For the second part, we note that

$$[X]_\sigma \vee [Y]_\sigma = [X \otimes Y]_\sigma \mapsto [(X \otimes Y')_\sigma] = [X' \otimes Y']_\sigma = [X']_\sigma \vee [Y']_\sigma.$$ 

Similarly we have also

$$x([X]_\sigma) = [x(X)]_\sigma = [(x(\psi_\top) : \psi_\top \in X)]_\sigma \mapsto [(x(X)]_\sigma = [[[x(\psi_\top)]_\sigma : \psi_\top \in X])_\sigma = [x(X')]_\sigma.$$ 

So the map between $\Psi_\top$ and $(\Psi \cup \Psi_\top) / \sigma / \theta$ is a homomorphism too. Condition (5.1) holds as is readily verified. This concludes the proof. \[\square\]
6.4 Continuous Information Algebras

The notion of approximation can be somewhat weakened. This leads to a generalization of the concept of compact information algebras. The present section is based on (Guan & Li, 2010a). The basic notion in this section is the way-below relation in an ordered set.

**Definition 6.5 Way-Below.** Let \( \Phi \) be a partially ordered set. For \( \phi, \psi \in \Phi \) we write \( \psi \ll \phi \) and say \( \psi \) is way-below \( \phi \), if for every directed set \( X \subseteq \Phi \), \( \phi \leq \vee X \) implies that there is an element \( \chi \in X \) such that \( \psi \leq \chi \).

Note that finite elements \( \phi \) in a compact information algebra satisfy the relation \( \phi \ll \phi \) (see Theorem 6.2 and Definition 6.2). The following lemma contains some well-known elementary results on the way-below relation.

**Lemma 6.5** Let \( \Phi \) be a partially ordered set. Then the following hold for \( \phi, \psi \in \Phi \)

1. \( \psi \ll \phi \) implies \( \psi \leq \phi \),
2. \( \psi \ll \phi \) and \( \phi \leq \chi \) imply \( \psi \ll \chi \),
3. \( \chi \leq \psi \) and \( \psi \ll \phi \) imply \( \chi \ll \phi \).
4. \( \chi \ll \psi \) and \( \psi \ll \phi \) imply \( \chi \ll \phi \).

The proof is left to the reader. We are of course interested in the way-below relation in case that \( (\Phi, D) \) is an information algebra, that is, \( \Phi \) is a semilattice. Then the way-below relation has some additional properties.

**Lemma 6.6** Let \( (\Phi, D) \) be an information algebra. Then

1. \( 0 \ll \phi \) for all \( \phi \in \Phi \).
2. \( \psi_1, \psi_2 \ll \phi \) implies \( \psi_1 \vee \psi_2 \ll \phi \) for all \( \psi_1, \psi_2 \in \Phi \).
3. The set \( \{ \psi \in \Phi : \psi \ll \phi \} \) is an ideal for all \( \phi \in \Phi \).
4. \( \psi \ll \phi \) if, and only if, for all \( X \subseteq \Phi \) such that \( \vee X \) exists and \( \phi \leq \vee X \), there is a finite subset \( F \) of \( X \) such that \( \psi \leq \vee F \).

**Proof.** (1) Let \( X \subseteq \Phi \) be a directed set, and \( \phi \leq \vee X \). Since \( X \) is non-empty, there is a \( \psi \in X \) and \( 0 \leq \psi \), hence \( 0 \ll \phi \).

(2) Assume \( \psi_1, \psi_2 \ll \phi \). Consider any directed set \( X \subseteq \Phi \) such that \( \phi \leq \vee X \). Then there exists elements \( \chi_1, \chi_2 \in X \) so that \( \psi_1 \leq \chi_1 \) and \( \psi_2 \leq \chi_2 \). Since \( X \) is directed, there is also an element \( \chi \in X \) so that \( \chi_1, \chi_2 \leq \chi \). But then, \( \psi_1 \vee \psi_2 \leq \chi_1 \vee \chi_2 \leq \chi \). This shows that \( \psi_1 \vee \psi_2 \ll \phi \).
6.4. CONTINUOUS INFORMATION ALGEBRAS

(3) Assume \( \psi \ll \phi \) and \( \chi \leq \psi \). Then by Lemma 6.5 (3) \( \chi \ll \phi \). Further let \( \psi_1 \ll \phi \) and \( \psi_2 \ll \phi \). By (2) just proved, \( \psi_1 \lor \psi_2 \ll \phi \). Hence \( \{ \psi \in \Phi : \psi \ll \phi \} \) is an ideal.

(4) Suppose first that \( \psi \ll \phi \). Let \( X \) be a subset of \( \Phi \) such that \( \lor X \) exists and \( \phi \leq \lor X \). Let \( Y \) be the set of all joins of finite subsets of \( X \). Then \( X \subseteq Y \) and \( \lor X \) is an upper bound for \( Y \). Let \( \chi \) be another upper bound of \( Y \). Then \( \lor X \leq \chi \). So \( \lor X \) is the supremum of \( Y \), \( \lor X = \lor Y \). Furthermore \( Y \) is a directed set. So there is an element \( \eta \in Y \) such that \( \psi \leq \eta \) and \( \eta = \lor F \) for some finite subset \( F \) of \( X \).

Conversely consider elements \( \psi, \phi \in \Phi \) such that condition (4) of the lemma holds. Let \( X \) be a directed subset of \( \Phi \) such that \( \lor X \) exists and \( \phi \leq \lor X \). There is by (4) a finite subset \( F \) of \( X \) such that \( \psi \leq \lor Y \). Since \( X \) is directed, there is a \( \chi \in X \) such that \( \lor Y \leq \chi \), hence \( \psi \leq \chi \). So \( \psi \ll \phi \). This concludes the proof of (4). \( \square \)

With the aid of the way-below relation, compact information algebras can be alternatively characterized.

**Theorem 6.14** If \((\Phi, D)\) is an information algebra, then the following are equivalent:

1. \((\Phi, D)\) is a \( D \)-compact information algebra with finite elements \( \Phi_f \).

2. \( \Phi \) is an algebraic lattice with finite elements \( \Phi_f \) and \( \forall x \in D, \forall \phi \in \Phi \)

\[
x(\phi) = \lor \{ \psi \in \Phi_f : x(\psi) \ll \phi \}.
\]  \hfill (6.15)

**Proof.** (1) \( \Rightarrow \) (2): By Theorem 6.2 \( \Phi \) as an ordered structure is an algebraic lattice, that is a complete lattice with finite elements \( \Phi_f \). Then condition (6.15) follows from strong density and Lemma 6.5 in the following way

\[
x(\phi) &= \lor \{ \psi \in \Phi_f : x(\psi) \leq \phi \} \\
&= \lor \{ \psi : \psi \ll \phi = x(\psi) \leq \phi \} \\
&= \lor \{ \psi : \psi \ll \phi = x(\psi) \ll \phi \} \\
&= \lor \{ \psi \in \Phi_f : x(\psi) \ll \phi \}.
\]

(2) \( \Rightarrow \) (1): Convergence holds, since \( \Phi \) is a complete lattice, density follows from (6.15) since \( \psi \ll \phi \) implies \( \psi \leq \phi \) and compactness follows from the lattice-theoretic finiteness. \( \square \)

Another important property of finite elements in a compact information algebra is given by the following theorem:

**Theorem 6.15** If \((\Phi, D)\) is a compact information algebra, then \( \forall \psi, \phi \in \Phi \), \( \psi \ll \phi \) implies that there is an element \( \chi \in \Phi_f \) so that \( \psi \leq \chi \leq \phi \).
CHAPTER 6. COMPACT AND CONTINUOUS ALGEBRAS

Proof. The set \( A_\phi = \{ \chi \in \Phi_f : \chi \leq \phi \} \) is directed and \( \phi \leq \lor A_\phi \). Then \( \psi \ll \phi \) implies the existence of an element \( \chi \in A_\phi \) so that \( \psi \leq \chi \). But \( \chi \leq \phi \). So \( \psi \leq \chi \leq \phi \) and \( \chi \in \Phi_f \). \( \square \)

A set of elements having the property that \( \psi \ll \phi \) implies the existence of a \( \chi \in S \) such that \( \psi \leq \chi \leq \phi \) is called separating. So the set of finite elements in a compact information algebra is separating.

We now introduce continuous information algebras and show that they are a generalization of compact ones.

Definition 6.6 Continuous Information Algebras. An information algebra \( (\Phi, D) \) is called continuous with basis \( B \subseteq \Phi \) if \( B \) is closed under join and \( B \) satisfies the following conditions:

1. Convergence: If \( X \subseteq B \) is directed, then \( \lor X \) exists in \( \Phi \).

2. (Weak) \( B \)-Density: \( \forall \phi \in \Phi \ \phi = \lor \{ \psi \in B : \psi \ll \phi \} \).

Note that in a compact information algebra \( (\Phi, D) \) the finite elements \( \Phi_f \) form a basis. So, a compact information algebra is also continuous with basis \( \Phi_f \). We shall present in a moment an example of a continuous information algebra which is not compact. So continuous information algebras present a genuine generalization of compact information algebras. The approximation by finite elements is replaced by an approximation of some more general elements in a basis \( B \). However, as in the case of compact information algebras, a stronger approximation condition is natural in an information algebra, corresponding to strong density in \( D \)-compact information algebras, expressing the possibility of approximation within a domain \( x \).

Definition 6.7 \( D \)-Continuous Information Algebras. An information algebra \( (\Phi, D) \) is called \( D \)-continuous with basis \( B \subseteq \Phi \) if it is continuous and if

(Strong) \( B \)-Density: \( x(\phi) = \lor \{ \psi \in B : \psi = x(\psi) \ll x(\phi) \} \) for all \( \phi \in \Phi \) and for all \( x \in D \).

Note that \( \psi \ll \phi \) does not imply \( x(\psi) \ll x(\phi) \). That is why in the strong density condition \( \psi = x(\psi) \ll x(\phi) \) instead of only \( \psi = x(\psi) \ll \phi \) is required.

If the information algebra \( (\Phi, D) \) is supported, then strong \( B \)-density implies weak \( B \)-density: In fact let \( \phi \in \Phi \), then there is a \( x \in D \) so that \( \phi = x(\phi) \). Then by the strong \( B \)-density:

\[
\phi = x(\phi) = \lor \{ \psi \in B : \psi = x(\psi) \ll \phi \} \\
\leq \lor \{ \psi \in B : \psi \ll \phi \} \leq \phi.
\]

This is weak \( B \)-density.
Again, a $D$-compact information algebra $(\Phi, D)$ with finite elements $\Phi_f$ is also $D$-continuous with basis $\Phi_f$. For later purposes we remark that both the sets $\{ \psi \in B : \psi \ll \phi \}$ and $\{ \psi \in B : \psi = x(\psi) \ll \phi \}$ are directed.

Here follows an example of a continuous information algebra, more examples will be given later (in particular see Section 6.6)

**Example 6.10** This example is due to (Guan & Li, 2010a). Let $\Phi = [0, 1]$ be the real interval between 0 and 1 and $D = \{0, 1\}$. Join is defined as maximum, $\phi \vee \psi = \max\{\phi, \psi\}$, and $0 = 0, 1 = 1$. Information extraction is defined as follows:

$$1(\phi) = \phi,$$
$$0(\phi) = \begin{cases} \phi & \text{if } \phi \in [0, 1/2], \\ 1/2 & \text{if } \phi \in (1/2, 1]. \end{cases}$$

Clearly these operators form an idempotent, commutative semigroup. We leave it to reader to verify that the operators are also existential quantifiers.

Any non-empty subset $X$ of $[0, 1]$ is in this example directed and sup $X$ exists always. The relation $\psi \ll \phi$ holds if either $0 < \psi < \phi$ and in particular if $\psi = \phi = 0$. As a basis we take $B = \Phi$. Then it can be verified that $x(\phi) = \vee\{ \psi \in B : \psi = x(\psi) \ll \phi \}$ holds both for $x = 0$ and $x = 1$. So it is a $\{0, 1\}$-continuous information algebra. But it is not compact: The only element satisfying $\phi \ll \phi$ is $\phi = 0$.

As this example shows, the basis of a continuous information algebra $(\Phi, D)$ may be $\Phi$ itself, although approximating elements simply by their lower bounds may not be all too interesting.

We have seen above that a compact or a $D$-compact information algebra is continuous or $D$-continuous. But the converse does not hold as the example above shows. Here follows a necessary and sufficient condition for a continuous information algebra to be compact.

**Theorem 6.16** A continuous, respectively $D$-continuous information algebra $(\Phi, D)$ is compact, respectively $D$-compact if, and only if, the set $\{ \phi \in \Phi : \phi \ll \phi \}$ is a basis for $(\Phi, D)$.

**Proof.** We know already that if $(\Phi, D)$ is compact, respectively $(D)$-compact, then it is continuous, respectively $D$-continuous, with basis $B = \Phi_f = \{ \phi \in \Phi : \phi \ll \phi \}$.

So, assume that $(\Phi, D)$ is continuous with basis $B = \{ \phi \in \Phi : \phi \ll \phi \}$. Then $(\Phi, D)$ is compact with $B$ as finite elements: Convergence is inherited from the continuous algebra $(\Phi, D)$. Density follows using Lemma 6.6:

$$\phi = \vee\{ \psi \in B : \psi \ll \phi \} \leq \vee\{ \psi \in \Phi : \psi \ll \psi \leq \phi \} = \vee\{ \psi \in B : \psi \leq \phi \} \leq \phi.$$
Strong density is shown similarly:

\[ x(\phi) = \bigvee \{ \psi \in B : \psi = x(\psi) \ll x(\phi) \} \]
\[ = \bigvee \{ \psi \in B : \psi = x(\psi) \ll \psi \leq x(\phi) \} \]
\[ = \bigvee \{ \psi \in B : \psi = x(\psi) \leq x(\phi) \} \]
\[ = \bigvee \{ \psi \in B : \psi = x(\psi) \leq \phi \} \]

Assume \( X \subseteq B \) directed, \( \phi \in B \) and \( \phi \leq \bigvee X \). Then \( \phi \ll \phi \) implies that there is a \( \psi \in X \) such that \( \phi \leq \psi \). This is the compactness condition.

Just as in a compact information algebra \((\Phi, D)\) the set \( \Phi \) is an algebraic lattice, it follows that in a continuous information algebra \((\Phi, D)\) the set \( \Phi \) is a \textit{continuous lattice}, namely a complete lattice such that for all \( \phi \in \Phi \)

\[ \phi = \bigvee \{ \psi \in \Phi : \psi \ll \phi \}. \] (6.16)

**Theorem 6.17** If \((\Phi, D)\) is an information algebra, then the following are equivalent:

1. \((\Phi, D)\) is continuous.
2. \(\Phi\) is a continuous lattice, that is \(\Phi\) is a complete lattice and (6.16) holds.

**Proof.** (1) \(\Rightarrow\) (2): Let \((\Phi, D)\) be a continuous information algebra with basis \(B\). We show first that \(\Phi\) is a complete lattice. Consider a non-empty subset \(X\) of \(\Phi\). Define \(Y\) to be the set of all elements in \(B\), which are way-below all elements in \(X\),

\[ Y = \{ \psi \in B : \psi \ll \phi, \forall \phi \in X \}. \]

The set \(Y\) is directed, since it is non-empty, and with \(\psi_1, \psi_2 \in Y\) also \(\psi_1 \lor \psi_2 \in Y\) (Lemma 6.6). Therefore \(\forall Y\) exists and is a lower bound of \(X\). Assume \(\psi\) to be another lower bound of \(X\). Then \(A_\psi = \{ \eta \in B : \eta \ll \psi \} \subseteq Y\), since \(\eta \ll \psi \leq \phi\) implies \(\eta \ll \phi\). From this we conclude that \(\psi = \bigvee A_\psi \leq \bigvee Y\), hence \(\forall Y\) is the infimum of \(X\). Since \(\Phi\) has further a top element \(1\) it follows from standard results of lattice theory, that \(\Phi\) is a complete lattice. Further, using density, we obtain for all \(\phi \in \Phi\),

\[ \phi = \bigvee \{ \psi \in B : \psi \ll \phi \} \leq \bigvee \{ \psi \in \Phi : \psi \ll \phi \} \leq \phi. \]

So \(\Phi\) is indeed a continuous lattice.

(2) \(\Rightarrow\) (1): If \(\Phi\) is a continuous lattice, we may take \(\Phi\) as a basis and then, with this basis, the information algebra \((\Phi, D)\) is continuous.

The proof above shows that if the information algebra \((\Phi, D)\) is continuous with basis \(B\), it is also continuous with basis \(\Phi\). So, as in the case of compact information algebra, continuous information algebras are not too interesting, since they are simply continuous lattices with extraction operators not interacting with continuity. More interesting is the case of \(D\)-continuous information algebras.
Theorem 6.18 If \((\Phi, D)\) is a supported information algebra, then the following are equivalent:

1. \((\Phi, D)\) is \(D\)-continuous.

2. \(\Phi\) is a complete lattice, and \(\forall x \in D, \forall \phi \in \Phi\).
   \[
x(\phi) = \bigvee \{ \psi \in \Phi : \psi = x(\psi) \ll x(\phi) \}. \tag{6.17}
\]

Proof. Assume first \((\Phi, D)\) to be supported and \(D\)-continuous. Then \((\Phi, D)\) is a continuous information algebra, hence \(\Phi\) is a continuous lattice (Theorem 6.17), hence a complete lattice. Further,

\[
x(\phi) = \bigvee \{ \psi \in B : \psi = x(\psi) \ll x(\phi) \}
\leq \bigvee \{ \psi \in \Phi : \psi = x(\psi) \ll x(\phi) \} \leq x(\phi),
\]

so (6.17) holds.

If \(\Phi\), on the other hand, is a complete lattice, then convergence holds with \(\Phi\) as a basis. Weak \(\Phi\)-density follows from strong \(\Phi\)-density (6.17), since \((\Phi, D)\) is supported. Hence \((\Phi, D)\) is continuous and (6.17) makes it \(D\)-continuous.

The following Theorem gives a necessary and sufficient condition for continuous information algebras to be \(D\)-continuous.

Theorem 6.19 A continuous information algebra \((\Phi, D)\) is \(D\)-continuous if, and only if, for all \(x \in D\) and any directed set \(X \subset \Phi\),

\[
x(\bigvee X) = \bigvee_{\phi \in X} x(\phi). \tag{6.18}
\]

Proof. Assume \((\Phi, D)\) to be a continuous information algebra and that (6.18) holds. Then \(\Phi\) is a complete lattice. Consider a \(\phi \in \Phi\) and let \(\phi' = x(\phi)\). Then by weak density \(\phi' = \bigvee \{ \psi \in \Phi : \psi \ll \phi' \}\), and \(\{ \psi \in \Phi : \psi \ll \phi' \}\) is a directed set. From this we deduce, using (6.18)

\[
x(\phi) = x(x(\phi)) = x(\bigvee \{ \psi \in \Phi : \psi \ll x(\phi) \})
= \bigvee \{ x(\psi) : \psi \ll x(\phi) \}.
\]

Let \(\eta = x(\psi)\) so that \(\eta = x(\eta) \leq \psi \ll x(\phi)\). From this it follows that \(\eta \ll x(\phi)\) and therefore,

\[
x(\phi) = \bigvee \{ \eta : \eta = x(\eta) = x(\psi), \psi \ll x(\phi) \}
\leq \bigvee \{ \eta : \eta = x(\eta) \ll x(\phi) \} \leq x(\phi).
\]

Hence we have \(x(\phi) = \bigvee \{ \eta : \eta = x(\eta) \ll x(\phi) \}\) and by Theorem 6.18 \((\Phi, D)\) is \(D\)-continuous.
Conversely, assume \((\Phi, D)\) to be \(D\)-continuous with basis \(B\), \(X \subseteq \Phi\) directed and \(x \in D\). For \(\phi \in X\) we have \(\phi \leq \bigvee X\), hence \(x(\phi) \leq x(\bigvee X)\) and therefore \(\bigvee_{\phi \in X} x(\phi) \leq x(\bigvee X)\). By strong \(B\)-density,
\[
x(\bigvee X) = \bigvee \{\psi \in B : \psi = x(\psi) \ll x(\bigvee X)\}.
\]
Now, \(\psi = x(\psi) \ll x(\bigvee X) \leq \bigvee X\) implies that there is a \(\phi \in X\) so that \(\psi \leq \phi\) and thus also \(\psi = x(\psi) \leq x(\phi)\). From this we conclude that \(x(\bigvee X) \leq \bigvee_{\phi \in X} x(\phi)\) and thus \(\bigvee_{\phi \in X} x(\phi) = x(\bigvee X)\) by strong \(B\)-density.

Note that Theorem 6.3 shows that (6.18) holds in case \((\Phi, D)\) is \(D\)-compact. This follows also from the second part of Theorem 6.19, since a compact information algebra is also continuous. On the other hand, if \((\Phi, D)\) is compact, hence continuous, and (6.18) holds, then Theorem 6.19 says that \((\Phi, D)\) is also \(D\)-continuous. Now, by Theorem 6.16, if \((\Phi, D)\) is compact, then \(\Phi_f = \{\phi \in \Phi : \phi \ll \phi\}\) is a basis for the \((\Phi, D)\). Since this algebra is \(D\)-continuous, by the same Theorem 6.16 we conclude that it is \(D\)-compact. So we have proved the following corollary.

**Corollary 6.1** A compact information algebra \((\Phi, D)\) is \(D\)-compact if, and only if, for all \(x \in D\) and any directed set \(X \subseteq \Phi\),
\[
x(\bigvee X) = \bigvee_{\phi \in X} x(\phi).
\]

Similar to the case of a \(D\)-compact information algebra \((\Phi, D)\), the subalgebra \((Fix_x, D)\) is \(D\)-continuous, if \((\Phi, D)\) is so. In fact, let \(B\) be a basis of \(\Phi\), then if \(X \subseteq Fix_x \cap B\) is directed, then by Theorem ??
\[
x(\bigvee X) = \bigvee_{\phi \in X} x(\phi) = \bigvee_{\phi \in X} \phi,
\]
so that \(x(\bigvee X) \in Fix_x\). Further, for any \(y \in D\), by strong density in \((\Phi, D)\),
\[
y(\phi) = y(x(\phi)) = (x \land y)(\phi)
\]
\[
= \bigvee \{\psi \in B : \psi = (x \land y)(\psi) \ll (x \land y)(\phi)\}
\]
\[
= \bigvee \{\psi \in B \cap Fix_x : \psi = y(\psi) \ll y(\phi)\}.
\]
This is strong density in the subalgebra \((Fix_x, D)\). Weak density in \(Fix_x\) follows from this for \(y = x\). So, \((Fix_x, D)\) is indeed \(D\)-continuous with base \(B \cap Fix_x\).

### 6.5 Labeled Continuous Algebras

What is the labeled version of a continuous information algebra? To examine this question, we consider the labeled version \((\Psi, D)\) of a \(D\)-continuous
6.5. LABELED CONTINUOUS ALGEBRAS

information algebra \((\Phi, D)\). As always, when we consider the labeled version of an information algebra, \(D\) must be a lattice (see Section 4.4). We remind that \(\Psi\) consists of all pairs \((\psi, x)\), where \(\psi \in \Phi\) and \(x = x(\psi)\), that is, \(x\) is a support of \(\psi\).

Assume that \(B\) is a basis of the continuous information algebra \((\Phi, D)\). Assume \(B\) to be a basis of the continuous algebra \((\Phi, D)\). Define \(B_x = \{ (\psi, x) : \psi \in B, x(\psi) = \psi \}\). We claim that \(B_x\) is a basis in \(\Psi_x\). In fact, if \(\phi, x \in B_x\), then \(\phi = \phi(\psi)\), hence \(\phi \leq \psi\) (see Section 2.1). So, \(B_x\) is closed under combination. Further, \((0, x)\) belongs to \(B_x\). Define \(\bar{B}_x = \bigcup_{x \in D} B_x\).

Then, even \(\bar{B}\) is closed under combination. In fact, let \((\phi, x) \in B_x\) and \((\psi, y) \in B_y\), then \(\phi, \psi \in B\) and \(x\) is a support of \(\phi\), \(y\) a support of \(\psi\). But then \(x \vee y\) is a support of \(\phi \vee \psi\) (see Section 2.1). So, \(\phi \vee \psi \in B\). If, further, \((B_x, D)\) is a subalgebra of \((\Phi, D)\), then, if \((\psi, x) \in B_x\) and \(y \leq x\), it follows that \(\psi \leq \chi\) (see Section 2.1). Therefore, \(B_x\) is also closed under projection and \((\bar{B}, D)\) is a subalgebra of \((\Psi, D)\).

Further, Lemma 6.2 holds also in the case of \(D\)-continuous information algebras:

**Lemma 6.7** Let \((\Phi, D)\) be a \(D\)-continuous information algebra. For all \(x \in D\) and for every subset \(X \subseteq \text{Fix}_x\), it holds that
\[
\bigvee_{\psi \in X} (\psi, x) = (\bigvee X, x).
\]

**Proof.** Since \((\text{Fix}_x, D)\) is \(D\)-continuous (see end of Section 6.4), \(\phi = \bigvee X\) belongs to \(\text{Fix}_x\), so \((\bigvee X, x) \in \Psi_x\). Obviously, \((\phi, x)\) is an upper bound of all \((\psi, x)\) with \(\psi \in X\). Let \((\chi, x)\) be another upper bound. Then \(\psi \leq \chi\) for all \(\psi \in X\), hence \(\phi \leq \chi\) and therefore \((\phi, x)\) is the least upper bound, \(\bigvee_{\psi \in X} (\psi, x) = (\phi, x)\). \(\square\)

Assume now that \(X \subseteq B_x\) is a directed set. Then \(X' = \{ \phi : (\phi, x) \in X \}\) is a directed set in \(\text{Fix}_x\). Therefore, by Lemma 6.7 \(\bigvee_{(\phi, x) \in X} (\phi, x) = (\bigvee X', x)\) belongs to \(\Psi_x\). This is a convergence property in \(\Psi_x\).

We claim that also a density property holds in \(\Psi_x\). For this purpose, we remind that \(\Psi\) is a semi lattice, and therefore the way-below relation \(\ll_x\) within \(\Psi_x\) is well defined for all \(x \in D\). Then, the following result is important.

**Lemma 6.8** Let \(\phi, \psi \in \Phi\) and \(x(\phi) = \phi, x(\psi) = \psi\). Then \(\psi \ll \phi\) if and only if \((\psi, x) \ll_x (\phi, x)\).
Proof. Assume \( \psi \ll \phi \) and \( x(\phi) = \phi \), \( x(\psi) = \psi \). Consider a directed set \( X \subseteq \Psi_x \). Then \( X' = \{ \phi : (\phi, x) \in X \} \) is directed too. Now, \((\phi, x) \leq \forall X \) implies \( \phi \leq \forall X' \) by Lemma 6.7. Then, there is a \( \chi \in X' \) such that \( \psi \leq \chi \). Note that \( x(\chi) = \chi \). Hence we see that \((\psi, x) \leq (\chi, x)\). So indeed \((\phi, x) \ll_x (\psi, x)\).

Conversely, assume \((\psi, x) \ll_x (\phi, x)\). Consider a directed set \( X \subseteq \Phi \) such that \( \phi \leq \forall X \). In a \( D \)-continuous information algebra we have \( x(\forall X) = \forall_{\phi \in X} x(\phi) \) (Theorem 6.19). Then \( \phi = x(\phi) \leq x(\forall X) = \forall_{\chi \in X} x(\chi) \). Therefore, by Lemma 6.7 \((\phi, x) \leq (\forall_{\chi \in X} x(\chi), x) = \forall_{\chi \in X} (x(\chi), x) \). Since the set \( \{(x(\chi), x) : \chi \in X \} \) is directed, there must then be a \( \chi \in X \) such that \((\psi, x) \leq (x(\chi), x)\). Then \( \psi = x(\psi) \leq x(\chi) \leq \chi \in X \). This proves that \( \psi \ll \phi \).

Now, using Lemma 6.7 and Lemma 6.8 strong density in \((\Phi, D)\),

\[
\forall \{ (\psi, x) \in B_x : (\psi, x) \ll_x (\phi, x) \} = \forall \{ (\psi, x) : \psi \in B, \psi = x(\psi), (\psi, x) \ll_x (\phi, x) \} = \forall \{ (\psi, x) : \psi \in B, \psi = x(\psi) \ll \phi = x(\phi) \} = \{ (\psi : \psi \in B, \psi = x(\psi) \ll \phi = x(\phi) \}, x \}
\]

This is the density property claimed above.

Finally, assume \((\psi, x) \ll_x (\phi, x)\). By Lemma 6.8 we have \( \psi \ll \psi \) and \( x \) is a support of both \( \psi \) and \( \phi \). If \( x \leq y \), then \( y \) is also a support of both elements (see Section 2.1). Therefore, again by Lemma 6.8 we have that \((\psi, x)^y = (\psi, y) \ll y (\phi, y) = (\phi, x)^y \). Conversely, assume that \( x \) is a support of \( \psi \) and \( \phi \) and \( x \leq y \). Then, if \((\psi, y) \ll y (\phi, y), \) Lemma 6.8 implies that \( \psi \ll \phi \), hence \((\psi, x) \ll_x (\phi, x)\). This exposes an important compatibility relation between the way-below relation in different domains \( \Psi_x \) and \( \Psi_y \).

We summarize these results in the following theorem.

**Theorem 6.20** Let \((\Phi, D)\) be a \( D \)-continuous information algebra with basis \( B \) and \((\Psi, D)\) the associated labeled information algebra. Then the following properties hold:

1. \( B_x \) is a basis in \( \Psi_x \), that is closed under combination and contains the neutral element \((0, x)\) of \( \Psi_x \).

2. If \((B, D)\) is a subalgebra of \((\Phi, D)\), then \((B, D)\) is a subalgebra of \((\Psi, D)\).

3. If \( X \subseteq B_x \) is directed, then \( \forall X \in \Psi_x \), for all \( x \in D \).

4. \((\phi, x) = \forall \{ (\psi, x) \in B_x : (\psi, x) \ll_x (\phi, x) \}, \) for all \( \phi, x \) in \( \Psi_x \).

5. If \( x \leq y \), then \((\psi, x) \ll_x (\phi, x)\) if and only if \((\psi, y) \ll_y (\phi, x)^y \).
This theorem serves as a base to define the concept of a labeled continuous information algebra.

**Definition 6.8** Labeled Continuous Information Algebra: A labeled information algebra \((\Psi, D)\) is called continuous, if there is for all \(x \in D\) a set \(B_x \subseteq \Psi_x\) (the basis in \(x\), closed under combination, containing \(0_x\) and satisfying the following conditions for all \(x \in D\):

1. **Convergence:** If \(X \subseteq B_x\) is directed, then \(\forall X \in \Psi_x\).
2. **Density:** For all \(\phi \in \Psi_x\), \(\phi = \vee\{\psi \in B_x : \psi \ll_x \phi\}\).
3. **Compatibility:** If \(d(\phi) = d(\psi) = x \leq y\), then \(\psi \ll_x \phi\) if and only if \(\psi \uparrow y \ll_y \phi \uparrow y\).

According to this definition and Theorem 6.20, the labeled information algebra \((\Psi, D)\) associated with a continuous information algebra \((\Phi, D)\) is itself continuous. We remark that, as in Theorem 6.17, it follows that \(\Psi_x\) is a continuous lattice for every \(x \in D\).

Conversely, let’s start with a labeled continuous information algebra \((\Psi, D)\) and consider its associated information algebra \((\Psi/\sigma, D)\). Is this algebra continuous too? An answer is given by Theorem 6.21 below. In order to prove this theorem we need two auxiliary results, which have some interest by themselves.

**Lemma 6.9** Let \((\Psi, D)\) be a labeled algebra. Then \(x([\psi]_\sigma) = [\psi]_\sigma \ll [\phi]_\sigma = x([\phi]_\sigma)\) in \(\Psi/\sigma\) implies \(\psi \ll_x \phi\) for any representants \(\psi\) and \(\phi\) of \([\psi]_\sigma\) and \([\phi]_\sigma\) with \(d(\psi) = d(\phi) = x\). Further if \(D\) has a top element \(\top\), and \((\Psi, D)\) is labeled continuous, then, for \(d(\psi) = d(\phi) = x\), \(\psi \ll_x \phi\) implies \([\psi]_\sigma \ll [\phi]_\sigma\).

**Proof.** Assume \(X \subseteq \Phi_x\) directed and \(\phi \leq \vee X\). Consider then \([\phi]_\sigma \leq \vee [X]_\sigma\) with \([X]_\sigma = \{[\chi]_\sigma : \chi \in X\}\). The set \([X]_\sigma\) is directed, therefore \([\psi]_\sigma \ll [\phi]_\sigma\) implies that there is a \(\eta \in X\) such that \([\psi]_\sigma \leq [\eta]_\sigma\), hence \(\psi \leq \eta\). This proves that \(\psi \ll_x \phi\).

For the second part, assume \(\psi \ll_\top \phi\) and consider a directed set \(X\) in \(\Phi/\sigma\). We may take as representatives of the elements \([\eta]_\sigma\) of \(X\) elements in \(\Phi_\top\). Let then \(X' = \{\eta \in \Phi_\top : [\eta]_\sigma \in X\}\). \(X'\) is still directed. Now, if \([\phi]_\sigma \leq \vee X\) and \(\phi\) is again a representative of \([\phi]_\sigma\) in \(\Phi_\top\), then also \(\phi \leq \vee X'\). Since \(\psi \ll_\top \phi\), there is an element \(\eta \in X'\) such that \(\psi \leq \eta\). But then \([\eta]_\sigma \in X\) and \([\psi]_\sigma \leq [\eta]_\sigma\). This shows that \([\psi]_\sigma \ll [\phi]_\sigma\). Now, if \(d(\psi) = d(\phi) = x\) and \(\psi \ll_x \phi\), then by the compatibility property \(\psi \uparrow \top \ll_\top \phi \uparrow \top\), hence \([\psi]_\sigma \ll [\phi]_\sigma\) as just proved.

**Lemma 6.10** Let \((\Psi, D)\) be a labeled continuous information algebra. If \(X \subseteq \Psi_y\) directed, then for all \(x \leq y \in D\),

\[
(\forall X)^{lx} = \vee X^{lx},
\]

where \(X^{lx} = \{\psi^{lx} : \psi \in X\}\).
Consider a $\psi_80$

**CHAPTER 6. COMPACT AND CONTINUOUS ALGEBRAS**

...Consider the representants of this set in $\Psi$...

This shows that (weak) density hold. Therefore, $(\Psi_\sigma, D)$ repeatedly Lemma 6.3 and Lemma 6.10 $X$.

Define $x$.

It is however needed to derive the strong density. By Theorem 6.19 it $X$.

Next consider any class $\{\psi\}_x \psi_\sigma$. We first show that $\Psi$.

Now we are in apposition to prove the following theorem.

**Theorem 6.21** Let $(\Psi, D)$ be a labeled continuous information algebra, and $D$ has a top element $\top$, then $(\Psi/\sigma, D)$ is $D$-continuous.

**Proof.** We first show that $\Psi/\sigma$ is a complete lattice. To this end consider any non-empty subset $X \subseteq \Psi/\sigma$. For any element $[\psi]_\sigma$ of $X$ we may take a representant $\psi$ in the top domain $\Psi_{\top}$, $d(\psi) = \top$. Let then $X' = \{\psi \in \Psi_{\top}: [\psi]_\sigma \in X\}$. But $\Psi_{\top}$ is a complete lattice, hence $\bigvee X'$ exists in $\Psi_{\top}$.

By Lemma 6.3 $\bigvee X' = \bigvee X \in \Psi/\sigma$. Since $\Psi/\sigma$ has a top element $[1_{\top}]_\sigma$, by standard results of lattice theory $\Psi/\sigma$ is a complete lattice.

Next consider any class $[\phi]_\sigma \in \Psi/\sigma$. The set $\{[\psi]_x : [\psi]_x \ll [\phi]_\sigma\}$ is directed. Consider the representants of this set in $\Psi_{\top}$: $[\psi_\top : [\psi_\sigma \ll [\phi]_\sigma]$ and also $\phi \in \Psi_{\top}$. Then, by Lemma 6.3

$\bigvee \{[\psi]_x : [\psi]_x \ll [\phi]_\sigma\} = \bigvee \{\psi \in \Psi_{\top} : [\psi]_\sigma \ll [\phi]_\sigma\}_\sigma$

This shows that (weak) density hold. Therefore, $(\Psi/\sigma, D)$ is a continuous information algebra.

So far we have not used the assumption that $D$ has a top element. It is however needed to derive the strong density. By Theorem 6.19 it is sufficient for this purpose to prove (6.18). So, consider a directed set $X \subseteq \Psi/\sigma$. For any $[\psi]_\sigma \in X$ we may select a representant $\psi_\sigma$ in $\Psi_{\top}$. Define $X' = \{\psi_\top : [\psi]_\sigma\}$. This set is still directed in $\Psi_{\top}$. Now, using repeatedly Lemma 6.3 and Lemma 6.10

$x(\bigvee X) = x(\bigvee \{[\phi]_\sigma : \phi \in X'\}) = x(\bigvee X') = (\bigvee X')_\sigma$

This proves that $(\Psi/\sigma, D)$ is $D$-continuous. □

Note that the existence of a top element in $D$ is required for $(\Psi/\sigma, D)$ to be $D$-continuous. The proof of the theorem above shows, that without this condition, at least $(\Psi/\sigma, D)$ is continuous. It remains so far an open
question, whether a labeled continuous information algebra \((\Psi, D)\) can be extended to a labeled continuous information algebra with a top domain, as it has been done for labeled compact information algebras (see Section 6.3). The problem is the extension of the compatibility condition to the new top domain.

A way to obtain labeled continuous information algebras is from continuous lattices as shown in the next section.

### 6.6 Algebras Induced by Algebraic Lattices

In Section 4.5 we have seen how a distributive lattice \(L\) induces a labeled information algebra \((\Psi, D)\) of maps \(\psi : \Omega_s \to L\) from \(s\)-tuples into the lattice. \(D\) denotes here the lattice of finite subsets of some index set \(I\). It is to be expected that the algebra \((\Psi, D)\) becomes compact or continuous if \(L\) is algebraic or continuous. In this section we show that this is indeed the case. However, note that for \(\psi, \phi \in \Psi_s\) for some finite subset \(s\) of an index set \(I\), we have \(\psi \leq \phi\) if, and only if, \(\psi(x) \geq_L \phi(x)\) for all \(x \in \Omega_s\) (see Section 4.5). Therefore, we need to consider the opposite order in \(L\), when we want algebraic or continuous lattices \(L\) to induce compact or continuous information algebras \((\Psi, D)\).

Let therefore \((L_{op}, \leq_{op})\) be the dual of lattice \((L, \leq_L)\), that is, with the opposite order, \(a \leq_{op} b\) if, and only if, \(b \leq_L a\). Now we assume \(L_{op}\) to be algebraic with \(F_{op}\) denoting its finite or compact elements. Note that \(L_{op}\) is then a complete lattice and for all \(a \in L\),

\[
a = \bigvee_{op}\{b \in F_{op} : b \leq_{op} a\}.
\]

Define \(\Psi_{s,f}\) to be the set of all maps \(\psi : \Omega_s \to F_{op}\) for \(s \in D\), and

\[
\Psi_f = \bigcup_{s \in D} \Psi_{s,f}.
\]  

(6.20)

We claim that the information algebra \((\Psi, D)\), induced by a lattice \(L\) is compact with finite elements \(\Psi_f\) as defined above, if the lattice \(L_{op}\) is algebraic.

In order to verify this, we check first that \(\Psi_f\) is closed under combination. Let \(\phi \in \Psi_{s,f}\) and \(\psi \in \Psi_{t,f}\), then, if we note that \(\wedge_L\) is \(\bigvee_{op}\), we obtain for all \(x \in \Omega_{s\cup t}\),

\[
(\phi \otimes \psi)(x) = \phi(x^{s}) \bigvee_{op} \psi(x^{t}) \in F_{op},
\]

since \(F_{op}\) is closed under join. So, indeed \(\phi \otimes \psi \in \Psi_f\).

Consider any non-empty subset \(X\) of \(\Psi_s\) and define for every \(x \in \Omega_s\) the set \(X_x = \{\psi^{(x)} : \psi \in X\}\). Since \(L_{op}\) is complete, the supremum \(\eta(x) = \bigvee_{op} X_x\) exists. So \(\eta \in \Psi_s\) and it is evident that \(\eta = \bigvee X\), so \(\Psi_s\) is
To verify density, consider \( \phi \in \Psi_s \). Then, for \( x \in \Omega_s \) define
\[
\eta(x) = \bigvee \{ \psi(x) : \psi \in \Psi_{s,f}, \psi(x) \leq_{op} \phi(x) \} \\
\leq \bigvee \{ b \in F_{op} : b \leq_{op} \phi(x) \} \leq_{op} \phi(x).
\]
Obviously, \( \eta = \bigvee \{ \psi \in \Psi_{s,f} : \psi \leq \phi \} \leq \phi \). Select for any \( x \in \Omega_s \) an element \( b_x \leq_{op} \phi(x) \), \( b_x \in F_{op} \) and define \( \psi(x) = b_x \). Then \( \psi \in \Psi_{s,f} \) and \( \psi \leq \phi \).
This shows that \( \eta = \phi \), which proves density.

Compactness of elements in \( \Psi_{s,f} \) in \( \Psi_s \) follows from the finiteness of \( F_{op} \) in \( L_{op} \): Let \( X \subseteq \Psi_{s,f} \) be directed and \( \phi \in \Psi_{s,f} \) so that \( \phi \leq \bigvee X \). Then we have \( \phi(x) \leq_{op} \bigvee_{\psi \in X} \psi(x) \) for all \( x \in \Omega_s \). Since the sets \( \{ \psi(x) : \psi \in X \} \) are directed too, there is a \( \chi_x \in X \) so that \( \psi(x) \leq_{op} \chi_x(x) \). Because \( \Omega_s \) is finite and \( X \) directed, there is a \( \chi \in X \) such that \( \chi_x \leq \chi \) for all \( x \in \Omega_s \), hence \( \psi(x) \leq_{op} \chi(x) \) or \( \psi \leq \chi \). This is compactness in \( \Psi_s \).

We have proved the following theorem

**Theorem 6.22** Let \((\Psi, D)\) be the labeled information algebra induced by the lattice \( L \), where the dual lattice \( L_{op} \) is algebraic with with finite elements \( F_{op} \). Then \((\Psi, D)\) is a labeled compact information algebra, whose finite elements are \( \Psi_f \) as given in (6.20).

If the index set \( I \) is finite, then \( I \in D \) is the top element of the lattice \( D \). According to Theorem 6.9, \((\Psi/\sigma, D)\) is then a \( D \)-compact information algebra.

Next we turn to information algebras induced by a lattice \( L_{op} \) whose dual lattice \( L_{op} \) is *continuous*. First we prove a lemma, which shows how the way-below relation in \( L_{op} \) is reflected in the semilattices \( \Psi_s \) of maps \( \psi : \Omega_s \to L_{op} \).

**Lemma 6.11** Let \((\Psi, D)\) the labeled information algebra induced by the lattice \( L \). Then \( \psi \ll_{s} \phi \) in \( \Psi_s \) if, and only if, \( \psi(x) \ll_{op} \phi(x) \) for all \( x \in \Omega_s \).

**Proof.** Consider first a directed subset \( X \) of \( \Psi_s \) and \( \phi \leq \bigvee X \). This means that \( \phi(x) \leq \bigvee_{\chi \in X} \chi(x) \) for all \( x \in \Omega_s \). The set \( \{ \chi(x) : \chi \in X \} \) is directed in \( L_{op} \). Therefore, \( \psi(x) \ll_{op} \phi(x) \) implies that there is a \( \chi_x \in X \) such that \( \psi(x) \leq_{op} \chi_x(x) \). By an argument as in the compactness proof above, there is then a \( \chi \in X \) so that \( \chi_x \leq \chi \) for all \( x \). Then we have \( \psi(x) \leq_{op} \chi(x) \), hence \( \psi \leq \chi \). This shows that \( \psi \ll_{s} \phi \).

Conversely, let \( X \) be directed in \( L_{op} \), select \( x_0 \in \Omega_s \), \( \phi \in \Psi_s \) and \( \phi(x_0) \leq \bigvee X \). Define
\[
Y = \{ \chi \in \Psi_s : \chi(x) = \phi(x) \text{ for } x \neq x_0, \chi(x_0) = a \in X \}.
\]
Then \( Y \) is a directed set in \( \Psi_s \) and \( \phi \leq \bigvee Y \). Now, \( \psi \ll_s \phi \) implies that there is a \( \chi \in Y \) such that \( \psi \leq \chi \), hence \( \psi(x_0) \leq \bigvee \chi(x_0) = a \in X \), and therefore we conclude that \( \psi(x_0) \ll_{\text{op}} \phi(x_0) \). Since \( x_0 \) is arbitrary this proves that \( \psi \ll_s \phi \) implies \( \psi \ll_{\text{op}} \phi(x) \) for all \( x \in \Omega_s \). \( \square \)

Let now \( B_{\text{op}} \) be a basis of the continuous lattice \( L_{\text{op}} \) and define

\[
B_s = \{ \psi \in \Psi_s : \Omega_s \to B_{\text{op}} \},
\]

and

\[
B = \bigcup_{s \in D} B_s.
\]

Clearly, \( B \) is closed under combination. Since \( L_{\text{op}} \) is a complete lattice, so is \( \Psi_s \) for all \( s \in D \). So convergence holds.

Consider further a \( \phi \in \Psi_s \) and define \( \eta = \bigvee \{ \psi \in B_s : \psi \ll_s \phi \} \). Using Lemma 6.11 and continuity of \( L_{\text{op}} \), we obtain for any \( x \in \Omega_s \),

\[
\eta(x) = \bigvee \{ \psi(x) \in B_{\text{op}} : \psi(x) \ll_{\text{op}} \phi(x) \}
\leq \bigvee \{ b \in B_{\text{op}} : b \ll_{\text{op}} \phi(x) \} \leq \phi(x).
\]

By an argument similar as in the proof of density above, we conclude that \( \eta(x) = \phi(x) \), so that \( \phi = \bigvee \{ \psi \in B_s : \psi \ll_s \phi \} \). This is density.

Note that if \( d(\psi) = s \) and \( s \subseteq t \), then \( \psi_{\text{op}} \) is defined by \( \psi_{\text{op}}(x) = \psi(x^{l_t}) \times 1_t = \psi(x_{l_t}) \). Therefore, finally, by Lemma 6.11 we have \( \psi \ll_s \phi \) if and only if \( \psi(x) \ll_{\text{op}} \phi(x) \) for all \( x \in \Omega_s \). By the same Lemma this holds if and only if \( \psi_{\text{op}} \ll_t \phi_{\text{op}} \). This is the compatibility relation.

We have proved the following theorem:

**Theorem 6.23** Let \( (\Psi, D) \) be the labeled information algebra induced by the lattice \( L \) where the dual lattice \( L_{\text{op}} \) is continuous with with basis \( B_{\text{op}} \). Then \( (\Psi, D) \) is a labeled continuous information algebra, with bases \( B_s \) as defined in (6.21).

Again if \( I \) is finite, from Theorem 6.21 it follows that \( (\Psi/\sigma, D) \) is a \( D \)-continuous information algebra.

If a labeled continuous information Algebra is induced by a lattice \( L \), then \( L_{\text{op}} \) must be continuous:

**Theorem 6.24** Let \( (\Psi, D) \) be a labeled information algebra, induced by a lattice \( L \). If \( (\Psi, D) \) is continuous, then \( L_{\text{op}} \) is a continuous lattice.

**Proof.** First, we show that \( L \) is complete. Consider any subset \( X \) of \( L \), select an \( s \in D \) and define \( \phi_a(x) = a \) for \( a \in X \) and for all \( x \in \Omega_s \). Then \( \phi = \bigvee_{a \in X} \phi_a \in \Psi_s \), since \( \Psi_s \) is a complete lattice. Therefore, \( \phi(x) = \)
\[ \forall a \in X \phi_a(x) = \vee_{op} X \in L. \] In addition, \( L \) has a top element, hence \( L \) and also \( L_{op} \) are complete.

Second, for all \( a \in L \) we must show that \( a = \vee \{ b \in L : b \ll_{op} a \} \). By the continuity of \( \Psi_s \) we have that \( \phi_a = \vee \{ \psi \in \Psi_s : \psi \ll_s \phi_a \} \). This implies that for all \( x \in \Omega_s \), using Lemma 6.11

\[
\begin{align*}
\alpha &= \phi_a(x) = \vee \{ \psi(x) : \psi \in \Psi_s : \psi \ll_s \phi_a \} \\
&= \vee \{ \psi(x) : \psi \in \Psi_s : \psi(x) \ll_{op} \phi_a(x) = a \} \\
&\leq \vee \{ b \in L : b \ll_{op} a \} \leq a.
\end{align*}
\]

This proves that indeed \( a = \vee \{ b \in L : b \ll_{op} a \} \) and \( L_{op} \) is a continuous lattice.

It remains an open question, whether \( L_{op} \) is algebraic, if the \( L \)-induced labeled information algebra \( (\Psi, D) \) is compact.

This is an example how properties of a lattice carry over to lattice induced information algebras.

### 6.7 Continuous Maps

Intuitively, maps of information should also be information. The examples of order preserving maps in Section 4.6 shows this already and this will be confirmed in this section. Consider two information algebras \((\Phi, D)\) and \((\Psi, E)\). To simplify notation we denote join and information extraction in both algebras by the same symbols, and the same holds for the order between elements. It will always be clear from the context, where the operation takes place.

When we consider maps \( f : \Phi \to \Psi \) between information elements, then we should require that less information in results in less information out. That is, if \( \psi \leq \phi \), then \( f(\psi) \leq f(\phi) \). Such maps are called order preserving. The set of all order preserving maps between \( \Phi \) and \( \Psi \) is denoted by \( [\Phi \to \Psi] \).

An immediate consequence for order preserving maps is that \( f(\phi) \vee f(\psi) \leq f(\phi \vee \psi) \). In particular semilattice homomorphisms are order preserving. Other examples of order-preserving maps of \( \Phi \) into itself are:

1. the identity mapping \( f(\phi) = \phi \),
2. the constant mapping \( f(\phi) = \psi \),
3. information extraction \( f(\phi) = x(\phi) \),
4. conditioning \( f(\phi) = \phi \vee \psi \) for a constant \( \psi \).

Note that \( f(0) \) is not always equal to \( 0 \), see the second and fourth example. Maps which satisfy \( f(0) = 0 \) are called strict. In Section 4.6 we have shown that order-preserving maps form an information algebra.
We are in particular interested in maps between continuous and compact information algebras. So, assume that \((\Phi, D)\) and \((\Psi, E)\) are continuous information algebras with bases \(B_\Phi\) and \(B_\Psi\) respectively. Here we are especially interested in maps \(f : \Phi \rightarrow \Psi\) which, in addition to order, preserve approximation:

**Definition 6.9** *Continuous Maps:* Let \((\Phi, D)\) and \((\Psi, E)\) be continuous information algebras with bases \(B_\Phi\) and \(B_\Psi\) respectively. Then a map \(f : \Phi \rightarrow \Psi\) is called continuous, if for all \(\phi \in \Phi\),

\[
f(\phi) = \bigvee \{f(\psi) : \psi \in B_\Phi, \psi \ll \phi\}.
\]

In continuous information algebras, \(\Phi\) and \(\Psi\) are continuous lattices and in lattice theory usually different definitions of continuous maps between continuous lattices are used. But these definitions are all equivalent to Definition 6.9 the following lemma shows (Davey & Priestley, 1990).

**Lemma 6.12** Let \((\Phi, D)\) and \((\Psi, E)\) be continuous information algebras with bases \(B_\Phi\) and \(B_\Psi\) respectively and \(f : \Phi \rightarrow \Psi\). Then the following are equivalent

1. For all \(\phi \in \Phi\),

\[
f(\phi) = \bigvee \{f(\psi) : \psi \in B_\Phi, \psi \ll \phi\}.
\]

2. For all \(\phi \in \Phi\),

\[
\{\psi \in B_\Psi : \psi \ll f(\phi)\} \subseteq \{\psi \in \Psi : \psi \leq f(\chi) \text{ for some } \chi \in B_\Phi, \chi \ll \phi\}.
\]

3. If \(X\) is a directed subset of \(\Phi\), then

\[
f(\bigvee X) = \bigvee_{\phi \in X} f(\phi).
\]

**Proof.** (1) \(\Rightarrow\) (2): We note first that (1) implies that \(f\) is order preserving. Assume now \(\psi \in B_\Psi, \psi \ll f(\phi)\). Then by (1)

\[
\psi \ll \bigvee \{f(\chi) : \chi \in B_\Phi, \chi \ll \phi\}.
\]

The set \(\{f(\chi) : \chi \in B_\Phi, \chi \ll \phi\}\) is directed since \(f\) is order preserving. Therefore it follows that there is a \(\chi \in B_\chi, \chi \ll \phi\) such that \(\psi \leq f(\chi)\). This proves (3).

(2) \(\Rightarrow\) (3): Consider a directed subset \(X\) of \(\Phi\) and define \(\phi = \bigvee X\). Consider a \(\psi \in B_\Psi, \psi \ll f(\phi)\). By (2) there is an element \(\chi \in B_\Phi\) such that \(\chi \ll \phi\) and \(\psi \leq f(\chi)\). Then there is an element \(\eta \in X\) such that \(\chi \leq \eta\). Hence we conclude that \(\psi \leq f(\chi) \leq f(\eta) \leq \bigvee f(X)\), because \(f\) is
order preserving. So \( f(\lor X) \) is an upper bound of the set \( A_f(\phi) = \{ \psi : \psi \in B_\Phi, \psi \ll f(\phi) \} \). From the continuity of \( \Psi \) we obtain
\[
f(\lor X) = \lor A_f(\phi) \leq \lor f(X).
\]
The inverse inequality between \( f(\lor X) \) and \( \lor f(X) \) is obvious. So, indeed \( f(\lor X) = \lor f(X) \).

(3) \implies (1): The set \( A_\phi = \{ \psi \in B_\Phi : \psi \ll \phi \} \) is directed and \( \phi = \lor A_\phi \).

Therefore, by (3)
\[
f(\phi) = f(\lor A_\phi) = \lor \{ f(\psi) : \psi \in B_\Phi, \psi \ll \phi \}.
\]

The second point of the Lemma above corresponds to the definition of a continuous map. It says that the map \( f(\phi) \) of any element of \( \Phi \) is determined, or can be approximated, by the maps of the elements of the basis, on the finite elements in the case of compact information algebra \((\Phi, D)\). In the third point of the Lemma the set on the left hand side contains the parts of \( f(\phi) \) which are in the basis \( B_\Phi \) of \( \Psi \) and point (3) asserts that to obtain these parts it is sufficient to input only elements of the basis \( B_\Phi \) of \( \Phi \). If both information algebras \((\Phi, D)\) and \((\Psi, E)\) are compact, then this means that to obtain a finite parts of \( f(\phi) \), it is sufficient to input a finite part of the argument \( \phi \):
\[
\{ \psi \in \Psi_f : \psi \leq f(\phi) \} \subseteq \{ \psi \in \Psi : \psi \leq f(\chi) \text{ for some } \chi \in \Phi_f, \chi \leq \phi \}.
\]

Let \([\Phi \to \Psi]\) denote the set of order-preserving maps between \( \Phi \) and \( \Psi \) and \([\Phi \to \Psi]_c\) the set of continuous maps. We have seen in the proof of Lemma 6.12 that \([\Phi \to \Psi]_c\) is a subset of \([\Phi \to \Psi]\). Assume now for the time being that \((\Phi, D)\) and \((\Psi, E)\) are compact information algebras. Consider an order-preserving map \( g \in [\Phi_f \to \Psi] \) and define a map \( f : \Phi \to \Psi \) by
\[
f(\phi) = \lor \{ g(\psi) : \psi \in \Phi_f, \psi \leq \phi \}.
\]

Note that \( f \) is an extension of \( g \) from \( \Phi_f \) to \( \Phi \). We claim that \( f \) is continuous. Fix \( \phi \in \Phi \) and consider \( \psi \in \Psi_f, \psi \leq f(\phi) \). The set \( \{ g(\chi) : \chi \in \Phi_f, \chi \leq \phi \} \) is directed in \( \Psi \). Therefore, there exists a \( \chi \in \Phi_f \) such that \( \chi \leq \phi \) and \( \psi \leq g(\chi) = f(\chi) \). Hence, by (3) of Lemma 6.12 \( f \) is continuous. This shows that the map \( f \mapsto f|\Phi_f \) which assigns to a \( f \in [\Phi \to \Psi]_c \) its restriction to \( \Phi_f \), that is, an element of \([\Phi_f \to \Psi]\), is onto \([\Phi_f \to \Psi]\). It is also one-to-one. If \( f, g \in f \in [\Phi \to \Psi]_c \) and \( f|\Phi_f = g|\Phi_f \), then
\[
f(\phi) = \lor \{ f(\psi) : \psi \in \Phi_f, \psi \leq \phi \}
= \lor \{ g(\psi) : \psi \in \Phi_f, \psi \leq \phi \} = g(\phi),
\]
thus \( f = g \). So the map \( f \mapsto f|\Phi_f \) is an order-isomorphism between \([\Phi \to \Psi]_c\) and \([\Phi_f \to \Psi]\) if \((\Phi, D)\) and \((\Psi, E)\) are compact algebras, or \( \Phi \) and \( \Psi \) algebraic lattices.
6.7. CONTINUOUS MAPS

The definition of the continuity of maps is a pure lattice-theoretic concept. Nevertheless it allows for some connections to information extraction and hence to information algebras. For instance, Theorem ?? shows that in the case of D-continuous information algebras, information extraction is a continuous operator. It is known from lattice theory that \([\Phi \to \Psi]_c\) is a continuous lattice under point-wise order, that is if \(f \leq_p g\) if and only if \(g(\phi) \leq f(\phi)\) for all \(\phi \in \Phi\). In particular, as \(f, g \in [\Phi \to \Psi]_c\) are order-preserving maps, \(f \vee g\) is still defined point-wise by \((f \vee g)(\phi) = f(\phi) \vee g(\phi)\) for all \(\phi \in \Phi\). Similarly, for \((x, y) \in D \times E\), \((x, y)(f)\) is defined by \((x, y)(f)(\phi) = y(f(x(\phi)))\). These are continuous maps:

**Theorem 6.25** Let \((\Phi, D)\) and \((\Psi, E)\) be D-continuous information algebras, \(f, g \in [\Phi \to \Psi]_c\) and \((x, y) \in D \times E\). Then \(f \vee g\) and \((x, y)(f) \in [\Phi \to \Psi]_c\).

**Proof.** That \(f \vee g\) is continuous follows straightforward from the continuity of \(f\) and \(g\). Let \(X \subseteq \Phi\) be a directed set. Then

\[
(f \vee g)(\vee X) = f(\vee X) \vee g(\vee X) = (\vee_{\phi \in X} f(\phi)) \vee (\vee_{\phi \in X} g(\phi))
\]

This part makes no use of strong density, hence holds also for continuous algebras \((\Phi, D)\) and \((\Psi, E)\) in general.

In order to show that \((x, y)(f)\) is continuous, let \(X \subseteq \Phi\) be a directed set. Then, using continuity of information extraction in both algebras \((\Phi, D)\) and \((\Psi, E)\), Theorem ??, as well as continuity of \(f\), we obtain

\[
(x, y)(f)(\vee X) = y(f(\vee X)) = y(f(\vee_{\phi \in X} x(\phi)))
\]

\[
= y(\vee_{\phi \in X} f(x(\phi))) = \vee_{\phi \in X} y(f(x(\phi)))
\]

So, \([\Phi \to \Psi]_c\) is closed under join and under the operators \((x, y) \in D \times E\), hence \(([\Phi \to \Psi]_c, D \times E)\) is a subalgebra of \(([\Phi \to \Psi], D \times E)\) and in particular itself an information algebra. This algebra is even continuous.

**Theorem 6.26** Let \((\Phi, D)\) and \((\Psi, E)\) be D-continuous information algebras. Then \(([\Phi \to \Psi]_c, D \times E)\) is a \(D \times E\)-continuous information algebra.

**Proof.** As noted above, by general lattice-theoretic results, \([\Phi \to \Psi]_c\) is a continuous lattice (Scott, 1971), hence \(([\Phi \to \Psi]_c, D \times E)\) a continuous information algebra (Theorem 6.17). According to Theorem ?? it remains only to show that information extraction commutes with join over directed sets. So, assume \(X \subseteq [\Phi \to \Psi]_c\) be a directed set of continuous functions,
take $\phi \in \Phi$ and $(x, y) \in D \times E$. Then, by the continuity of the map $\vee X$ and of the operators in $D$ and $E$,

$$(x, y)(\vee X)(\phi) = y((\vee X)(x(\phi))) = y(\vee_{f \in X} f(x(\phi))) = \vee_{f \in X} y(f(x(\phi))) = \vee_{f \in X} (x, y)(f)(\phi).$$

So it holds indeed that $(x, y)(\vee X) = \vee_{f \in X} (x, y)(f)$. This proves that $([\Phi \to \Psi], D \times E)$ is a $D \times E$-continuous information algebra. □

What is not clear so far, is how the basis in $([\Phi \to \Psi], D \times E)$ is related to the bases of $(\Phi, D)$ and $(\Psi, E)$. It has been shown in (Kohlas, 2003a) that $([\Phi \to \Psi], D \times E)$ is $D \times E$-compact if $(\Phi, D)$ and $(\Psi, E)$ are $D$- and $E$-compact respectively. In this case the finite elements of $[\Phi \to \Psi]$ are well defined in terms of the finite elements of $\Phi$ and $\Psi$ (Kohlas, 2003a).

### 6.8 Cartesian Closure

To conclude this chapter we are going to consider the following categories of information algebras:

1. The category $\text{IA}$ has as objects information algebras $(\Phi, D)$ and as morphisms monotone maps $\Phi \to \Psi$.

2. The category of $D$-continuous information algebras $\text{ContIA}$ has as objects $D$-continuous information algebras and as morphisms continuous maps $\Phi \to \Psi$.

3. The category of $D$-compact information algebras $\text{CompIA}$ has as objects $D$-compact information algebras and as morphisms continuous maps $\Phi \to \Psi$.

The category $\text{CompIA}$ is a subcategory of $\text{ContIA}$, which itself is a subcategory of $\text{IA}$. We are going to show that all these categories are Cartesian closed. The (weakly) continuous and compact information algebras are essentially continuous or algebraic lattices with an additional family of operation. These lattices are known to be Cartesian closed. Therefore we shall not dwell on these cases.

To remind: A category $\mathcal{C}$ is Cartesian closed, if it satisfies the following three conditions:

1. The category $\mathcal{C}$ has a terminal object: There is an object $T \in \mathcal{C}$ such that there is exactly one morphism from any object to $T$.

2. The category $\mathcal{C}$ has finite products: For any pair of objects $A, B \in \mathcal{C}$, there is an object $A \times B$ and morphisms $p_A : A \times B \to A$ and $p_B : A \times B \to B$, such for any object $C$ and for pair of morphisms $f_1 : C \to A$ and $f_2 : C \to B$ there is a morphism $f : C \to A \times B$ so that $p_A \circ f = f_1$ and $p_B \circ f = f_2$. 


3. The category $\mathbf{C}$ has exponentials: For any pair of objects $B, C \in \mathbf{C}$, there is an object $C^B$ and a morphism $\text{eval} : C^B \times B \to C$ such that for every morphism $f : A \times B \to C$ there is a unique morphism $\lambda f : A \to C^B$ so that $\text{eval} \circ (\lambda f, id_B) = f$.

We have seen in Section 5.3 that information algebras have direct products with homomorphisms as morphisms. This holds in the same way with monotone maps as morphisms as we shall show. With homomorphisms there is no exponential, whereas with monotone maps there is. We shall directly examine the category $\text{ContIA}$, the results for the other categories can then be obtained in the same way.

The system $(\Phi, D)$, where $\Phi$ and $D$ consist each of one element is clearly an information algebra, and as a finite one, also compact and continuous. This is the terminal object in all of the categories $\text{IA}, \text{CompIA}$ and $\text{ContIA}$.

**Theorem 6.27** The Cartesian product $(\Phi \times \Psi, D \times E)$ of a $D$-continuous and an $E$-continuous information algebras $(\Phi, D)$ and $(\Psi, E)$ under component-wise join and meet respectively and also component-wise information extraction, is the categorical direct product of $(\Phi, D)$ and $(\Psi, E)$ in $\text{ContIA}$.

**Proof.** In fact, we have already shown in Section 5.3 that $(\Phi \times \Psi, D \times E)$ is an information algebra. Let then $B_\Phi$ and $B_\Psi$ be bases in $\Phi$ and $\Psi$ respectively. Obviously $B_\Phi \times B_\Psi$ is closed under join and contains the bottom element $(0_\Phi, 0_\Psi)$, where $0_\Phi$ and $0_\Psi$ are the bottom elements of $\Phi$ and $\Psi$ respectively. We claim that $B_\Phi \times B_\Psi$ is a basis of $\Phi \times \Psi$. Let $X \subseteq B_\Phi \times B_\Psi$ be a directed set and define $X_1 = \{ \phi \in B_\Phi : \exists \psi \in B_\Psi \text{ so that } (\phi, \psi) \in X \}$. $X_2$ is define similarly as the set of elements in $B_\Psi$ obtained from $X$. Both $X_1$ and $X_2$ are clearly directed. Then $(\vee X_1, \vee X_2)$ is an upper bound of $X$, and it is obviously its supremum. So $\vee X = (\vee X_1, \vee X_2)$ exists in $\Phi \times \Psi$. This is the convergence property.

We have $(\phi_1, \psi_1) \ll (\phi_2, \psi_2)$ if, and only if, $\phi_1 \ll \phi_2$ and $\psi_1 \ll \psi_2$, the $\ll$-relation taken in $\Phi \times \Psi$, $\Phi$ and $\Psi$ respectively. Consider $(\phi, \psi) \in \Phi \times \Psi$. Then

$$\bigvee \{(\phi', \psi') \in B_\Phi \times B_\Psi : (\phi', \psi') \ll (\phi, \psi)\}$$

$$= (\vee \{ \phi' \in B_\Phi : \phi' \ll \phi \}, \vee \{ \psi' \in B_\Psi : \psi' \ll \psi \})$$

$$= (\phi, \psi).$$

This shows that $\Phi \times \Psi$ is a continuous lattice.

If $(x, y) \in D \times E$, then we obtain in the same way

$$\bigvee \{(\phi', \psi') \in B_\Phi \times B_\Psi : (\phi', \psi') = (x, y)(\phi', \psi') \ll (x, y)(\phi, \psi)\}$$

$$= (\vee \{ \phi' \in B_\Phi : \phi' = x(\phi) \ll x(\phi) \}, \vee \{ \psi' \in B_\Psi : \psi' = y(\psi) \ll y(\phi) \})$$

$$= (x(\phi), y(\psi)) = (x, y)(\phi, \psi).$$
So strong density holds too. This proves that \((\Phi \times \Psi, D \times E)\) is a \(D \times E\)-continuous information algebra.

Define the projections \(p_\Phi : \Phi \times \Psi \to \Phi\) and \(p_\Psi : \Phi \times \Psi \to \Psi\) by \(p_\Phi(\phi, \psi) = \phi\) and \(p_\Psi(\phi, \psi) = \psi\). This are evidently continuous maps. Let then finally \((\Lambda; F)\) be an \(F\)-continuous information algebra and \(f_1\) and \(f_2\) be continuous maps \(f_1 : \Lambda \to \Phi\) and \(f_2 : \Lambda \to \Psi\). Then we define \(f = (f_1, f_2)\) as a map from \(\Lambda\) to \(\Phi \times \Psi\). It is continuous, since its components \(f_1\) and \(f_2\) are so.

Then clearly \(p_\Phi \circ f = f_1\) and \(p_\Psi \circ f = f_2\). This completes the proof. □

If \((\Phi, D)\) and \((\Psi, E)\) are \(D\)-respectively \(E\)-compact algebras, then \((\Phi \times \Psi, D \times E)\) is \(D \times E\)-compact, and its finite elements are \(\Phi f \times \Psi f\), since \((\phi, \psi) \ll (\phi, \psi)\) exactly if \(\phi \ll \phi\) and \(\psi \ll \psi\). Also, if \(f_1\) and \(f_2\) are monotone, then so is \(f = (f_1, f_2)\). Hence we have the corollary.

**Corollary 6.2** The Cartesian product \((\Phi \times \Psi, D \times E)\) of a \(D\)-compact and an \(E\)-compact information algebra \((\Phi, D)\) and \((\Psi, E)\) under component-wise join and meet respectively and component-wise information extraction, is the categorial direct product of \((\Phi, D)\) and \((\Psi, E)\) in \(\text{CompIA}\).

The Cartesian product \((\Phi \times \Psi, D \times E)\) of information algebras \((\Phi, D)\) and \((\Psi, E)\) under component-wise join and meet respectively, is the categorial direct product of \((\Phi, D)\) and \((\Psi, E)\) in \(\text{IA}\).

Next we show that the \(D \times E\)-continuous information algebra \(([\Phi \to \Psi], D \times E)\) of continuous maps between \(D\)-respectively \(E\)-continuous information algebras \((\Phi, D)\) and \((\Psi, E)\) is the exponential \((\Psi, E)^{([\Phi \to \Psi])}\).

**Theorem 6.28** The \(D \times E\)-continuous information algebra \(([\Phi \to \Psi], D \times E)\) is the exponential \((\Psi, E)^{([\Phi \to \Psi])}\) in \(\text{ContIA}\).

**Proof.** We have shown in Section 6.7 that \(([\Phi \to \Psi], D \times E)\) is a \(D \times E\)-continuous information algebra. We define the morphism \(\text{eval} : [\Phi \to \Psi] \times \Phi \to \Psi\) for \(f \in [\Phi \to \Psi]\) and \(\phi \in \Phi\) by

\[
\text{eval}(f, \phi) = f(\phi)
\]

The map \(\text{eval}\) is continuous.

Let \(f : \Lambda \times \Phi \to \Psi\) be a continuous map into the \(F\)-continuous information algebra \((\Psi, E)\). Then we define a map \(\lambda f : \Lambda \to [\Phi \to \Psi]\) for \(\chi \in \Lambda\) and \(\phi \in \Phi\)

\[
\lambda f(\chi) = f(\chi, \phi).
\]

The map \(\lambda f\) is continuous if \(f\) is. In fact, let \(X\) be a directed set in \(\Lambda\). Then we have for \(\phi \in \Phi\),

\[
\lambda f(\bigvee X)(\phi) = f(\bigvee X, \phi) = f(\bigvee_{\chi \in X}(\chi, \psi)) = \bigvee_{\chi \in X} f(\chi, \psi) = \bigvee_{\chi \in X} \lambda f(\chi)(\phi).
\]
Thus we see that $\lambda f(\lor X) = \lor_{\chi \in X} \lambda f(\chi)$.

Now finally for $(\chi, \phi) \in \Lambda \times \Phi$, we obtain that $\text{eval} \circ (\lambda f, \text{id}_\Phi)(\chi, \phi) = \text{eval}(\lambda f(\chi), \phi) = \lambda f(\chi)(\phi) = f(\chi, \phi)$. So indeed $\text{eval} \circ (\lambda f, \text{id}_\Phi) = f$. This completes the proof. □

Note that $([\Phi \to \Psi]_c, D \times E)$ is a $D \times E$-compact information algebra, if $(\Phi, D)$ and $(\Psi, E)$ are $D$-compact respectively $E$-compact. Further, if $(\Phi, D)$ and $(\Psi, E)$ are any information algebras, then in Section 4.6 we have seen that $([\Phi \to \Psi], D \times E)$ is an information algebra of monotone maps. Then eval and $\lambda f$ as defined in the proof above are also monotone. So the following corollary holds:

**Corollary 6.3** The $D \times E$-compact information algebra $([\Phi \to \Psi]_c, D \times E)$ is the exponential $(\Psi, E)^{(\Phi, D)}$ in $\text{CompIA}$.

The information algebra $([\Phi \to \Psi], D \times E)$ is the exponential $(\Psi, E)^{(\Phi, D)}$ in $\text{IA}$.

This shows that the categories $\text{IA}$, $\text{CompIA}$ and $\text{ContIA}$ are all Cartesian closed.
Chapter 7

Representation Theory

7.1 Relational Algebras

The archetypical example of an information algebra (in the labeled version) is the relational algebra of database theory. This is essentially a subset-algebra. The question we address here is whether any information algebra can be related to or, more precisely, embedded into a set algebra, in particular a relational algebra (in its labeled version). We remind that an information algebra \((\Phi, D)\) has a labeled version if and only if it is supported and the operators \(D\) form a lattice, see Section 4.4, especially Theorem 4.2. So, as far as we consider labeled versions of information algebras, we shall assume that \(D\) is a lattice.

As a preparation we present an abstract version of relational information algebras, see Section 4.4. For this purpose the notion of tuples will be generalized in a first step. This is based on (Kohlas, 2003a). If \(D\) is a lattice, then a set \(T\) with two operations \(d : T \rightarrow D\) and \([\cdot] : T \times D \rightarrow T\), defined for \(x \leq d(f)\), is a tuple system over \(D\), if it satisfies the following conditions for \(f, g \in T\) and \(x, y \in D\):

1. If \(x \leq d(f)\), then \(d(f[x]) = x\).
2. If \(x \leq y \leq d(f)\), then \(f[y][x] = f[x]\).
3. If \(d(f) = x\), then \(f[x] = f\).
4. If \(d(f) = x, d(g) = y\) and \(f[x \wedge y] = g[x \wedge y]\), then there exists a \(h \in T\) such that \(d(h) = x \vee y\) and \(h[x] = f, h[y] = g\).
5. If \(d(f) = x\) and \(x \leq y\), then there exists a \(g \in T\) such that \(d(g) = y\) and \(g[x] = f\).

The elements of the set \(T\) are called tuples, \(d(f)\) is the domain of tuple \(f\) and \(f[x]\) the projection of tuple \(f\) to domain \(x\). Obviously, ordinary tuples as used in ordinary relational algebra (Section 4.4), form a tuple system in
this sense over the lattice of finite subsets of an index set \( I \). We hint further at the fact that the equivalence classes of the equivalence relations \( \equiv_x \) for \( x \) in a lattice \( D \) associated with a Galois connection as discussed in Section 4.3 form a tuple system over \( D \) (see Section 4.3). The domain of a class \([m]_x\) is simply \( d([m]_x) = x \) and projection is defined as \([m]_y[x] = [m]_x \) if \( x \leq y \). The verification of the five conditions above is straightforward.

Using a tuple system \((T, D)\) the notion of a relational algebra can be generalized: A relation over \( x \in D \) is a set of tuples such that \( d(f) = x \) for all \( f \in R \). The domain of \( f \) can be assigned to \( R \), that is, we define \( d(R) = d(f) \). For a relation \( R \) and a \( x \leq d(R) \), the (relational) projection of \( R \) is defined as follows:

\[
\pi_x(R) = \{ f[x] : f \in R \}.
\]

The (relational) join of a relation \( R \) over \( x \) and a relation \( S \) over \( y \) is defined by

\[
R \Join S = \{ f \in T : d(f) = x \lor y, f[x] \in R, f[y] \in S \}.
\]

Finally, for \( x \in D \), the full relation over \( x \) is \( E_x \), the set of all tuples over \( x \). If \( R_T \) is the set of all relations of the tuple system \((T, D)\), then \((R_T, D)\) is a labeled information algebra with relational join as combination, with relational projection as projection and with \( E_x \) and the empty sets as vacuous information \( 0_x \) and contradictions \( 1_x \) in domain \( x \). The algebraic system \((R_T, D)\) is called a generalization relational algebra.

**Theorem 7.1** Let \((T, D)\) be a tuple system over \( D \). Then, the algebraic system \((R_T, D)\) is a labeled information algebra.

This is proved just as in the case of an ordinary relational algebra (see Section 4.4). Of course an ordinary relational algebra is a model of such a generalized relational algebra. As a further example we may consider the relational algebra with the equivalence classes \([m]_x\) in a Galois connection as tuples. In fact, the information algebra of closed model sets (see Section 4.3) is embedded into this relational algebra. If every model is closed, as for instance in propositional logic, then the algebra is the relational algebra of tuples \([m]_x\) itself.

Further, if \((\Psi, D)\) is a labeled information algebra, then \((\Psi, D)\) is also a tuple system over \( D \) with domain function \( d \) and projection \( \downarrow \) as tuple projection (Kohlas, 2003a). Therefore, the relational information algebra \((R_\Psi, D)\) over \( D \) can be considered. By the map \( \phi \mapsto \uparrow \phi = \{ \psi \in \Psi_x : \psi \geq \phi \} \) for \( \phi \in \Psi_x \) the labeled algebra \((\Psi, D)\) is embedded into \((R_\Psi, D)\). More explicitly, we have

\[
\begin{align*}
\phi \otimes \psi & \mapsto \uparrow \phi \Join \uparrow \psi, \\
\phi & \mapsto \pi_x(\uparrow \phi).
\end{align*}
\]
This is a first of a remarkable series of results presented in this chapter, which show how information algebras, labeled or not are isomorphic too or embedded into information algebras of sets; sets of abstract tuples in the case of labeled information algebras. The observation just made shows that the following theorem holds

**Theorem 7.2** Let \((\Psi, D)\) be a labeled information algebra. This algebra is embedded into the generalized relational algebra \((R_{\Psi}, D)\).

This is a *representation* of a labeled information algebra by a subalgebra of a relational algebra. This result is also related to a certain topology associated with information algebras (see Chapter 8). There it will also be shown that a similar result holds also for any information algebra \((\Phi, D)\). In many particular cases alternative interesting representations of information algebras, labeled or not, as subset or relational algebras are possible, as will be shown in the remaining sections of this chapter.

### 7.2 Atomic Algebras

Another way to relate information algebras to relational algebras is using the concept of atoms. Atoms are the maximal, the most informative elements of an information algebra. We consider first labeled information algebras \((\Psi, D)\).

**Definition 7.1** Labeled Atoms: An element \(\alpha \in \Psi\) with \(d(\alpha) = x\) is called an atom on domain \(x\), if

1. \(\alpha \neq 1_x\),
2. \(\forall \psi \in \Psi, d(\psi) = x\) and \(\alpha \leq \psi\) imply either \(\alpha = \psi\) or \(\psi = 1_x\).

In a pure order-theoretic view, atoms should be called co-atoms, since atoms usually are smallest elements. In the view of information order however the notion of an atom is justified, since it will turn out below, that a piece of information may be identified by the set of atoms implying the information. Also in instances of information algebras, atoms usually are what one would expect in the given framework. As an example, the one-tuple relations in (classical or generalized) relational algebras are atoms in this labeled information algebra. By abuse of language we say that the tuples themselves are the atoms of a relational algebra. Of course, information algebras may have no atoms. The following lemma collects a few elementary results on atoms, which are used frequently:

**Lemma 7.1** Let \((\Psi, D)\) be a labeled information algebra. Then
1. If \( \alpha \) is an atom on \( x \), \( \psi \in \Psi \) with \( d(\psi) = x \), then either \( \alpha \otimes \psi = \alpha \) or \( \alpha \otimes \psi = 1_x \).

2. If \( \alpha \) is an atom on \( x \), \( y \leq x \), then \( \alpha \downarrow y \) is an atom on \( y \).

3. If \( \alpha, \beta \) are atoms on \( x \), then either \( \alpha = \beta \) or \( \alpha \otimes \beta = 1_x \).

For the simple proofs we refer to (Kohlas, 2003a).

Let \( \text{At}(\Psi_x) \) denote the set of atoms of \( \Psi \) with domain \( x \), \( \text{At}(\Psi) \) the set of all atoms and for \( \psi \in \Psi \) with \( d(\psi) = x \),

\[
\text{At}(\psi) = \{ \alpha \in \text{At}(\Psi_x) : \psi \leq \alpha \}.
\]

Note that \( \text{At}(\psi) \) is a subset of \( \text{At}(\Psi_x) \). If \( \alpha \in \text{At}(\psi) \) we say also that \( \alpha \) is contained in \( \psi \). For instance, all tuples of a relation \( R \) are contained in \( R \). In this example any relation \( R \) is a lower bound of all the atoms it contains, in fact it is the infimum of \( \text{At}(R) \). Moreover, the set of any tuples has the relation \( R \) it forms as an infimum. We capture these properties in the following definition:

**Definition 7.2 Labeled Atomic Algebras:**

1. A labeled information algebra \((\Psi, D)\) is called atomic, if for all \( \psi \neq 1_x \) the set set \( \text{At}(\psi) \) is not empty.

2. A labeled information algebra \((\Psi, D)\) is called atomic composed, if for all \( \psi \in \Psi \), \( \psi \neq 1_x \),

\[
\psi = \wedge \text{At}(\psi).
\]

3. A labeled information algebra \((\Psi, D)\) is called atomic closed, if it is atomic composed and if for all \( x \in D \) and for every subset \( A \subseteq \text{At}(\Psi_x) \) the infimum \( \wedge A \) exists and belongs to \( \Psi \).

So, a relational algebra is atomic closed. The labeled information algebras of affine spaces, convex sets and polyhedra (see Section 3.4) are all atomic composed, but not closed. Note that an atomic composed algebra is atomic. Further in an atomic information algebra the atoms \((\text{At}(\Psi), D)\) form a tuple system (Kohlas, 2003a). This is contained in the tuple system \((\Psi, D)\). The associated relational algebra is \((R_{\text{At}(\Psi)}, D)\). More on this in the following theorem:

**Theorem 7.3** If \((\Psi, D)\) is an atomic, labeled information algebra, then the map \( \psi \mapsto \text{At}(\psi) \) is a homomorphism of \((\Psi, D)\) into the relational algebra \((R_{\text{At}(\Psi)}, D)\).
7.2. ATOMIC ALGEBRAS

Proof. We have to show that
\[
\begin{align*}
\phi \otimes \psi & \mapsto \At(\phi) \Join \At(\psi), \\
\phi^{1x} & \mapsto \pi_x(\At(\phi)), \\
0_x & \mapsto \At(\Psi_x), \\
1_x & \mapsto \emptyset.
\end{align*}
\]
Consider \(\phi, \psi \in \Psi\) such that \(d(\phi) = x\) and \(d(\psi) = y\). Assume \(\alpha \in \At(\phi \otimes \psi)\). Then \(\phi \otimes \psi \leq \alpha\), hence \(\phi \leq \alpha\) and also \(\phi = \phi^{1x} \leq \alpha^{1x}\). But \(\alpha^{1x}\) is an atom on \(x\), hence \(\alpha^{1x} \in \At(\phi)\). Similarly we find that \(\alpha^{1x} \in \At(\psi)\), thus \(\alpha \in \At(\phi) \Join \At(\psi)\).

Conversely, assume \(\alpha \in \At(\phi) \Join \At(\psi)\). Then \(d(\alpha) = x \lor y\) and \(\phi \leq \alpha^{1x} \leq \alpha\). In the same way we conclude that \(\psi \leq \alpha^{1y} \leq \alpha\). This implies \(\phi \otimes \psi \leq \alpha\) or \(\alpha \in \At(\phi \otimes \psi)\). Thus, we see that \(\At(\phi \otimes \psi) = \At(\phi) \Join \At(\psi)\).

Further let \(\alpha \in \At(\phi^{1x})\), such that \(d(\alpha) = x\) and \(\phi^{1x} \leq \alpha\). Suppose \(d(\phi) = y\). Then \(\phi \leq \phi \otimes \alpha\). Suppose first that \(\phi \otimes \alpha = 1_y\). Then we obtain
\[
\alpha = \phi^{1x} \otimes \alpha = (\phi \otimes \alpha)^{1x} = 1_y^{1x} = 1_x.
\]
But this is a contradiction, so \(\phi \otimes \alpha \neq 1_y\). Since \(\Phi\) is atomic this implies \(\At(\phi \otimes \alpha) \neq \emptyset\). Let \(\beta \in \At(\phi \otimes \alpha)\) such that \(d(\beta) = y\) and \(\phi \leq \phi \otimes \alpha \leq \beta\). This shows that \(\beta \in \At(\phi)\). But it holds also that
\[
\alpha = (\phi \otimes \alpha)^{1x} \leq \beta^{1x}.
\]
Since \(\alpha\) and \(\beta^{1x}\) are both atoms on \(x\), it follows \(\alpha = \beta^{1x}\), which shows that \(\alpha \in \pi_x(\At(\phi))\).

Conversely, consider \(\beta \in \pi_x(\At(\phi))\). This mean that there is a \(\gamma \in \At(\phi)\) such that \(\beta = \gamma^{1x}\). Then \(\phi \leq \gamma\), hence \(\phi^{1x} \leq \gamma^{1x} = \beta\). But this means that \(\beta \in \At(\phi^{1x})\). So \(\At(\phi^{1x}) = \pi_x(\At(\phi))\).

The last two propositions are evident. \(\square\)

Now, if \((\Psi, D)\) is atomic composed, then the map \(\psi \mapsto \At(\psi)\) is an embedding of \((\Psi, D)\) into the relational algebra \((\mathcal{R}_{\At(\psi)}, D)\). So, this is a further representation of a labeled information algebras by a subalgebra of a relational algebra. If \((\Psi, D)\) is atomic closed, then the map \(\psi \mapsto \At(\psi)\) is even an isomorphism, provided for sets of atoms \(X, Y \subseteq \At(\Psi_x)\), \(X \neq Y\) we have that \(\land X \neq \land Y\). In this case \((\Psi, D)\) is a labeled Boolean information algebra, as will be seen in Sections 7.3 and 7.4.

We may be somewhat more precise about the relation between the domains \(\At(\Psi_x)\) of the labeled subsets. Assume \(x \leq y\). A map \(\tau : \At(\Psi_x) \to \mathcal{P}(\At(\Psi_y))\) is called a refining, if

1. \(\forall \alpha \in \At(\Psi_x), \tau(\alpha) \neq \emptyset\),
2. If \((\alpha \neq \beta)\), then \(\tau(\alpha) \cap \tau(\beta) = \emptyset\),
3. $\bigcup_{\alpha \in \text{At}(\Psi_x)} \tau(\alpha) = \text{At}(\Psi_y)$.

Such a mapping is called a *refining* of $\text{At}(\Psi_x)$, since it splits the elements of $\text{At}(\Psi_x)$ and induces a partition $\{\tau(\alpha) : \alpha \in \text{At}(\Psi_x)\}$ in $\text{At}(\Psi_y)$. Here is the natural refining of atoms to a finer partition:

**Theorem 7.4** If $(\Psi, D)$ is an labeled atomic information algebra, then for $x, y \in D$ with $x \leq y$ the map

$$\tau(\alpha) = \{\beta \in \text{At}(\Psi_y) : \beta \downarrow x = \alpha\}.$$ 

between $\text{At}(\Psi_x)$ and $\mathcal{P}(\text{At}(\Psi_y))$ is a refining.

**Proof.** First we note that, since $\Psi$ is atomic, there is a $\beta \in \text{At}(\Psi_y)$. But then $\beta \downarrow x \leq (\alpha \uparrow y) \downarrow x = \alpha$, hence $\beta \in \tau(\alpha)$. Thus, $\forall \alpha \in \text{At}(\Phi)$, the set $\tau(\alpha)$ is not empty.

Next let $\alpha' \neq \alpha'' \in \text{At}(\Psi_x)$. Assume $\beta \in \tau(\alpha') \cap \tau(\alpha'')$, i.e. $\beta \downarrow x = \alpha'$ and $\beta \downarrow x = \alpha''$. But this implies $\alpha' = \alpha''$, contrary to the assumption. So $\tau(\alpha') \cap \tau(\alpha'') = \emptyset$.

Finally we have

$$\bigcup_{\alpha \in \text{At}(\Psi_x)} \tau(\alpha) \subseteq \text{At}(\Psi_y).$$

Consider $\beta \in \text{At}(\Psi_y)$. Then $\beta \downarrow x \in \text{At}(\Psi_x)$, hence $\beta \in \tau(\beta \downarrow x)$. Thus we have

$$\bigcup_{\alpha \in \text{At}(\Psi_x)} \tau(\alpha) = \text{At}(\Psi_y).$$

This concludes the proof that $\tau$ is a refining of $\text{At}(\Psi_x)$.

Let $\mathcal{F} = \{\text{At}(\Psi_x) : x \in D\}$. Then we may consider $\mathcal{F}$ as a family of sets of atoms, with the order $\text{At}(\Psi_x) \leq \text{At}(\Psi_y)$ if $x \leq y$. The family $\mathcal{F}$ is in fact a lattice with $\text{At}(\Psi_x) \land \text{At}(\Psi_y) = \text{At}(\Psi_{x \land y})$ and $\text{At}(\Psi_x) \lor \text{At}(\Psi_y) = \text{At}(\Psi_{x \lor y})$. Hence $\mathcal{R}_{\text{At}(\Psi)}$ can be seen as an information algebra of subsets from the family $\mathcal{F}$ and $(\Phi, D)$ is homomorphic to this algebra.

If $D$ has a top element $\top$, then all $\text{At}(\Psi_x)$ for $x < \top$ are essentially partitions of $\text{At}(\Psi)$ and $x \leq y$ implies $\text{At}(\Psi_x) \leq \text{At}(\Psi_y)$ in the partition order. If $D$ has no greatest element, then we may always go over to the labeled information algebra $(\Psi \cup \Psi_\top, D \cup \{\top\})$ by adjoining a top domain (see Section 4.4) and apply the same argument. So $\mathcal{F}$ is essentially a sublattice of the partition lattice of the universe $\Psi_\top$. Then we may replace the labeled information algebra $(\mathcal{R}_{\text{At}(\Psi)}, D)$ by the isomorphic labeled information algebra $(\mathcal{R}_{\text{At}(\Psi)}, \mathcal{F})$, which is an algebra of subsets of $\text{At}(\Psi_\top)$. This is an enhancement of Theorem 7.3 concerning the representation of atomic labeled information algebras by set algebras.

Similar results hold for information algebras. Let now $(\Phi, D)$ be an information algebra.
7.2. ATOMIC ALGEBRAS

Definition 7.3 **Atoms:** An element \( \alpha \in \Phi \) is called an atom, if

1. \( \alpha \neq 1 \),
2. \( \forall \phi \in \Phi, \alpha \leq \phi \) implies either \( \alpha = \phi \) or \( \phi = 1 \).

Let again \( \text{At}(\Phi) \) denote the set of atoms of \((\Phi, D)\), and

\[
\text{At}(\phi) = \{ \alpha \in \text{At}(\Phi) : \phi \leq \alpha \}.
\]

As above, we say that \( \phi \) contains atom \( \alpha \) if \( \alpha \in \text{At}(\phi) \). These atoms have similar properties as labeled atoms, Lemma 7.1,

Lemma 7.2 Let \((\Phi, D)\) be an information algebra. Then

1. If \( \alpha \) is an atom, then for all \( \phi \in \Phi \) either \( \alpha \lor \phi = \alpha \) or \( \alpha \lor \phi = 1 \).
2. If \( \alpha, \beta \) are atoms, then either \( \alpha = \beta \) or \( \alpha \lor \beta = 1 \).
3. If \( \alpha, \beta \) are atoms, \( x \in D \), then either \( x(\alpha) = x(\beta) \) or \( x(\alpha) \lor x(\beta) = 1 \).

Proof. (1) and (2) are similar to (1) and (3) of Lemma 7.1 and the proof is left to the reader.

In order to prove (3) note that \( \alpha = \beta \) implies \( x(\alpha) = x(\beta) \). So assume \( \alpha \neq \beta \). From (1) it follows that \( x(\alpha) \lor \beta = \beta \) or \( x(\alpha) \lor \beta = 1 \) according to whether \( x(\alpha) \leq \beta \) or not. Similarly, \( \alpha \lor x(\beta) = \alpha \) or \( = 1 \) according to whether \( x(\beta) \leq \alpha \) or not. Assume then that \( x(\alpha) \lor \beta = 1 \). In this case it follows that \( x(\alpha) \lor x(\beta) = 1 \). But then also \( \alpha \lor x(\beta) = 1 \). So assume \( x(\alpha) \lor \beta = \beta \), that is \( x(\alpha) \leq \beta \). Then it holds also that \( \alpha \lor x(\beta) = \alpha \), hence \( x(\beta) \leq \alpha \). But this implies \( x(\alpha) = x(\beta) \).

In the multivariate subset-algebra (see Section 3.4) the one-point-sets of \( \Omega_I \) are atoms, if the index set \( I \) belongs to \( D \), otherwise there are no atoms. For instance, if \( I = \omega \) and \( D \) is the lattice of finite subsets of \( \omega \). The one-element sets consisting of points \( x \in \mathbb{R}^\omega \) would be atoms, if \( \omega \) would belong to \( D \). But this is not the case, so there are not atoms. In the former case, where \( I \) is a finite set, any subset of \( \Omega_I \) is the infimum of the atoms it contains and the infimum of any set of atoms is an element of the algebra. So, similar to labeled algebras we capture this in the following definition:

Definition 7.4 **Atomic Algebras:**

1. An information algebra \((\Phi, D)\) is called atomic, if for all \( \phi \in \Phi, \phi \neq 1 \) implies that \( \text{At}(\phi) \) is not empty.
2. An information algebra \((\Phi, D)\) is called atomic composed, if for all \( \phi \in \Phi, \phi \neq 1 \),

\[
\phi = \land \text{At}(\phi).
\]
3. An information algebra \((\Phi, D)\) is called atomic closed, if it is atomic composed and if for every subset \(A \subseteq \text{At}(\Phi)\) the infimum \(\wedge A\) exists and belongs to \(\Phi\).

The following result is used later:

**Lemma 7.3** Let \((\Phi, D)\) be an atomic information algebra. Then, if \(\alpha, \beta\) are atoms such that \((x \land y)(\alpha) = (x \land y)(\beta)\), there is an atom \(\gamma\) such that \(x(\alpha) = x(\gamma)\) and \(y(\beta) = y(\gamma)\).

**Proof.** Assume \((x \land y)(\alpha) = (x \land y)(\beta)\) and define \(\phi = x(\alpha) \lor y(\beta)\). Then \(x(\phi) = x(x(\alpha) \lor y(\beta)) = x(\alpha) \lor (x(y(\beta)) = x(\alpha) \lor (x \land y)(\beta) = x(\alpha)\) and similarly \(y(\phi) = y(\beta)\). Suppose \(\phi = 1\). Then \(x(\phi) = x(\alpha) = 1\), which is excluded, since \(\alpha\) is an atom. So \(\phi \neq 1\) and \(\text{At}(\phi)\) is not empty. Select a \(\gamma \in \text{At}(\phi)\) such that \(\phi \leq \gamma\). But then \(x(\phi) = x(\alpha) \leq x(\gamma)\). From (3) of Lemma 7.2 it follows that \(x(\alpha) = x(\gamma)\). Similarly, we obtain \(y(\beta) = y(\gamma)\).

In the set of atoms \(\text{At}(\Phi)\) the notions of \(x\)-cylindrification for every \(x \in D\) can be introduced as follows: Define the relation \(\alpha \equiv_x \beta\) if \(x(\alpha) = x(\beta)\). This is an equivalence relation, whose equivalence classes will be denoted by \([\alpha]_x\). Then, if \(A\) is a set of atoms,

\[c_x(A) = \bigcup_{\alpha \in A} [\alpha]_x = \{\beta \in \text{At}(\Phi) : \beta \equiv_x \alpha \in A\}.\]

is called the \(x\)-cylindrification or \(x\)-saturation of \(A\). With intersection as combination and cylindrification as extraction \((\{\text{At}(\phi) : \phi \in \Phi\}, D)\) becomes an information algebra.

**Theorem 7.5** If \((\Phi, D)\) is an atomic information algebra, then \((\{\text{At}(\phi) : \phi \in \Phi\}, D)\) is an information algebra with intersection as combination and cylindrification as extraction.

**Proof.** With intersection as join, \((\{\text{At}(\phi) : \phi \in \Phi\}\) is a join-semilattice since \(\text{At}(\phi) \cap \text{At}(\psi) = \text{At}(\phi \lor \psi)\), with \(\text{At}(\Phi)\) and \(\emptyset\) as 0 and 1 elements respectively.

We show that \(c_x(c_y(S)) = c_{x \land y}(S)\) for any subset \(S\) of \(\text{At}(\Phi)\). Assume first that \(\alpha \in c_x(c_y(S))\). This means that there exist \(\beta, \gamma \in \text{At}(\Phi)\), such that \(x(\alpha) = x(\beta), y(\beta) = y(\gamma)\) and \(\gamma \in S\). But then we obtain that \((x \land y)(\alpha) = (x \land y)(\beta) = (x \land y)(\gamma)\) and this means that \(\alpha \in c_{x \land y}(S)\). On the other hand assume \(\alpha \in c_{x \land y}(S)\) such that there is an atom \(\gamma\) such that \((x \land y)(\alpha) = (x \land y)(\gamma)\) and \(\gamma \in S\). By Lemma 7.3 there is an atom \(\beta \in \text{At}(\Phi)\), such that \(x(\alpha) = x(\beta)\) and \(y(\beta) = y(\gamma)\). But this implies that \(\alpha \in c_x(c_y(S))\). So \(c_x(c_y(S)) = c_{x \land y}(S)\), which implies that the cylindrifications \(c_x\) for \(x \in D\) form a commutative, idempotent semigroup.
As a saturation of an equivalence relation or partition each cylindrification $c_x$ is also an existential quantifier. This concludes the proof. \(\square\)

As in the case of a labeled atomic information algebra, the map $\phi \mapsto \text{At}(\phi)$ is a homomorphism.

**Theorem 7.6** If $(\Phi, D)$ is an atomic information algebra, then the map $\phi \mapsto \text{At}(\phi)$ is a homomorphism of $(\Phi, D)$ into the subset algebra $(\mathcal{P}(\text{At}(\Phi)), D)$.

*Proof.* We have to show the following

1. $\phi \lor \psi \mapsto \text{At}(\phi) \cap \text{At}(\psi)$,
2. $x(\phi) \mapsto c_x(\text{At}(\phi))$,
3. $0 \mapsto \text{At}(\Phi)$,
4. $1 \mapsto \emptyset$.

(1) This has been noted already in the proof of the previous Theorem 7.5.

(2) Assume first $\alpha \in c_x(\text{At}(\phi)$ such that $x(\alpha) = x(\beta)$ for some atom $\beta \in \text{At}(\phi)$. Then $\phi \leq \beta$ implies $x(\beta) = x(\alpha) \leq x(\phi)$, hence $\alpha \in \text{At}(x(\phi))$. If, on the other hand, $\alpha \in \text{At}(x(\phi))$, then $x(\phi) \leq \alpha$. We have $\phi \leq x(\alpha) \lor \phi$. Assume that $x(\alpha) \lor \phi = 1$. Then $x(\alpha) \lor x(\phi) = x(x(\alpha) \lor \phi) = x(\alpha \lor x(\phi)) = 1$. But this implies $\alpha \lor x(\phi) = 1$ which contradicts the assumption $\alpha \in \text{At}(x(\phi))$. So, $x(\alpha) \lor \phi \neq 1$. Since the information algebra is atomic there is an atom $\beta \in \text{At}(x(\alpha) \lor \phi)$. Then $\phi \leq x(\alpha) \lor \phi \leq \beta$ implies $\beta \in \text{At}(\phi)$ and $\beta \lor (x(\alpha) \lor \phi) = \beta$. From this we conclude that

$$x(\beta) = x(\beta) \lor x(x(\alpha) \lor \phi) = x(\beta) \lor x(\alpha) \lor x(\phi) = x(\alpha) \lor x(\beta).$$

Then by Lemma 7.2 (3) $x(\alpha) = x(\beta)$, hence $\alpha \in c_x(\text{At}(\phi))$. This proves 2. above.

3. and 4. are evident. \(\square\)

If $(\Phi, D)$ is atomic composed, the map $\phi \mapsto \text{At}(\phi)$ is an embedding of $(\Phi, D)$ into the information algebra $(\text{At}(\Phi), D)$. This shows that an atomic composed information algebra is isomorph to some subset-information algebra, where combination is represented by set-intersection and extraction by set-saturation relative to a family of equivalence relations $\equiv_x$ for $x \in D$. These equivalence relations form a semilattice under the usual order of equivalence relations. This is of course similar to the case of atomic labeled information algebras.

**Theorem 7.7** If $(\Phi, D)$ is an atomic information algebra, then for $x, y \in D$, $x \leq y$ implies $\equiv_y \leq_p \equiv_x$ and $\equiv_{x \lor y} = \equiv_x \lor_p \equiv_y$, where $\leq_p$ denotes the usual order between equivalence relations or partitions and $\lor_p$ the corresponding join.
Proof. If \( x \leq y \), then \( y(\alpha) = y(\beta) \) implies \( x(\alpha) = x(\beta) \), therefore \([\alpha]_y \subseteq [\alpha]_x \) or \( \equiv_y \leq \equiv_x \). Consider a pair \( \alpha \equiv x \land y \beta \), i.e. \((x \land y)(\alpha) = (x \land y)(\beta)\). Then, by Lemma 7.3 there is a \( \gamma \in At(\Phi) \) such that the sequence \( \alpha, \gamma, \beta \) of atoms, which satisfies \( x(\alpha) = x(\gamma) \), \( y(\gamma) = y(\beta) \). But this means \( \equiv_x \land y \equiv x \lor \equiv y \). □

This or a similar schema will be seen below to repeat itself in different contexts.

7.3 Finite Boolean Algebras

An information algebra of sets is Boolean. Besides the combination, represented by the supremum or join, there exists the infimum or meet between any two pieces of information, and the resulting lattice is distributive. There exists also the complement or negation of any piece of information. So, let’s consider more generally Boolean information algebras. For Boolean algebras there exists an elaborated representation theory (Stone duality), (Davey & Priestley, 1990). How does this extend to Boolean information algebras?

Here is the definition of a Boolean information algebra:

Definition 7.5 Boolean Information Algebra: An information algebra \((\Phi, D)\) is called Boolean, if \( \Phi \) is a Boolean algebra.

There are many examples of Boolean information algebras, besides set-algebras. Quantor algebras based on Boolean algebras (see Section 4.1), information algebras related to propositional or predicate logic (see Section 4.3), cylindric algebras (Henkin et al., 1971) for instance are Boolean information algebras. For later reference, we note here a few often used results regarding Boolean information algebras.

Lemma 7.4 If \((\Phi, D)\) is a Boolean information algebra, then

1. \( \phi \leq \psi \) if and only if \( \psi^c \leq \phi^c \),
2. \( \phi \lor \psi^c = 1 \) if and only if \( \phi \geq \psi \).
3. \( \phi = x(\phi) \) if and only if \( \phi^c = x(\phi^c) \),
4. \( \phi = x(\phi) \) and \( \psi = x(\psi) \) imply \( \phi \land \psi = x(\phi \land \psi) \),
5. \( \phi \land x(\phi) = x(\phi) \).
6. For all \( x \in D \), \( x(\phi \land \psi) = x(\phi) \land x(\psi) \).

Proof. The proof (1) to (5) is either straightforward or known from Boolean algebras.
In order to prove (6) let \( \eta = \phi \land \psi \). Then \( \eta \leq \phi, \psi \), hence \( x(\eta) \leq x(\phi), x(\psi) \) and thus \( x(\eta) \leq x(\phi) \land x(\psi) \). So \( x(\eta) \) is a lower bound of \( x(\phi) \) and \( x(\psi) \).

Let \( t \) be another lower bound of \( x(\phi) \) and \( x(\psi) \). Then \( x(\phi) \lor t^c = 1 \) and \( x(\psi) \lor t^c = 1 \) by 2. of the lemma. Thus we obtain \( 1 = x(x(\phi) \lor t^c) = x(\phi) \lor x(t^c) \). But this implies \( \phi \lor x(t^c) = 1 \). Similarly we obtain \( \psi \lor x(t^c) = 1 \). If we take the meet of these two equations, we obtain

\[
1 = (\phi \lor x(t^c)) \land (\psi \lor x(t^c))
\]

As above we conclude from this that \( x(\eta) \lor t^c = 1 \), which implies \( x(\eta) \geq t \). Therefore, \( x(\eta) \) is the greatest lower bound of \( x(\phi) \) and \( x(\psi) \). \( \square \)

In this section we first examine the case of finite Boolean information algebras. So \( \Phi \) is a finite Boolean algebra. As is well known from the theory of finite Boolean algebras, for all \( \phi \in \Phi \),

\[
\phi = \land At(\phi).
\]

So, the information algebra \((\Phi, D)\) is atomic composed and even atomic closed, since any set of atoms \( A \) is finite and hence \( \land A \in \Phi \). Then by Section 7.2 the map \( \phi \mapsto At(\phi) \) is an information algebra isomorphism between \((\Phi, D)\) and \((P(At(\Phi)), D)\). It is even a Boolean algebra isomorphism. This proves in view of Theorem 7.6 the following theorem:

**Theorem 7.8 Representation Theorem for Finite Boolean Algebras.** Let \((\Phi, D)\) be a finite Boolean information algebra. Then the map \( \phi \mapsto At(\phi) \) is an isomorphism between \((\Phi, D)\) and the set information algebra \((P(At(\Phi)), D)\), such that

\[
\begin{align*}
\phi \lor \psi & \mapsto At(\phi) \cap At(\psi), \\
\phi \land \psi & \mapsto At(\phi) \cup At(\psi), \\
\phi^c & \mapsto At^c(\phi), \\
x(\phi) & \mapsto c_x(At(\phi)), \\
0 & \mapsto At(\Phi) \\
1 & \mapsto \emptyset,
\end{align*}
\]

So, any finite Boolean information algebra is isomorphic to the set information algebra of its atoms, with combination replaced by set-intersection and extraction by saturations or cylindrifications with respect to a certain family of equivalence relations \( \equiv_x \) in \( At(\Phi) \) for \( x \in D \). According to Theorem 7.7 this family of equivalence relations form a semilattice such that \( \equiv_{x \land y} = \equiv_x \lor \equiv_y \). Conversely, let \( S \) is a finite set together with an indexed
family of equivalence relations \( \equiv_x \), \( x \in D \) of \( S \), such that \( \equiv_x \lor p \equiv_y \) belongs to the family. Then the family of equivalence relations forms a semilattice, which in turn induces an order into \( D \) such that \( D \) becomes a semilattice and \( \equiv_x \lor p \equiv_y = \equiv_x \land y \). Then \( (P(S), D) \) form a Boolean information algebra with intersection as information join and \( x \)-saturation \( c_x \) relative to the equivalence relation \( \equiv_x \) as information extraction.

There is also a labeled version of Boolean information algebras.

**Definition 7.6 Labeled Boolean Information Algebra:** A labeled information algebra \((\Psi, D)\) is called Boolean, if

1. \( \forall x \in D, \Psi_x \) is a Boolean algebra,
2. \( \forall x, y \in D \) and \( \forall \phi, \psi \in \Psi \) with \( d(\phi) = d(\psi) = x \leq y \)
   \[ (\phi \land \psi) \otimes 0_y = (\phi \otimes 0_y) \land (\psi \otimes 0_y). \]

The second condition in this definition is necessary to guarantee that the information algebra \((\Psi/\sigma, D)\) is Boolean. Conversely, the labeled version of any supported Boolean information algebra satisfies this condition. Note that \( \Psi \) itself is not a Boolean algebra. The ordinary as well as the generalized relational algebras are labeled Boolean information algebras.

We consider again only finite labeled Boolean information algebras in this section. This means that for all \( x \in D \) \( \Psi_x \) are finite Boolean algebras. Again, from the theory of Boolean algebras it is known that \((\Psi, D)\) is atomic closed, and in particular for all \( \psi \in \Psi \),

\[ \psi = \wedge \text{At}(\psi). \]

so, according to Theorem 7.3 the following theorem holds:

**Theorem 7.9 Representation Theorem for Finite Labeled Boolean Algebras.** Let \((\Psi, D)\) be a finite labeled Boolean information algebra. Then the map \( \phi \mapsto \text{At}(\phi) \) is an isomorphism between \((\Psi, D)\) and the subset information algebra \((P(\text{At}(\Psi)), D)\), such that

\[
\begin{align*}
\phi \lor \psi & \mapsto \text{At}(\phi) \bowtie \text{At}(\psi), \\
\psi^x \mapsto & \pi_x(\text{At}(\psi)), \\
\psi^c & \mapsto \text{At}^c(\psi), \\
e & \mapsto \text{At}(\Psi_x), \\
z & \mapsto \emptyset,
\end{align*}
\]

So, a finite, labeled Boolean information algebra is isomorphic to the relational algebra of its atoms. Combination is represented by relational join.
7.4. BOOLEAN INFORMATION ALGEBRAS

and projection by relational projection. This, of course is the labeled version of the corresponding result for finite Boolean information algebras. As shown in Section 7.2, the family of sets $At(\Psi_x)$, for $x \in D$ forms a sublattice of partitions.

For general Boolean information algebras, the situation is only slightly more involved as the following section shows.

7.4 Boolean Information Algebras

Let now $(\Phi, D)$ be a Boolean information algebra. The key concept for representing such an algebra by a subset information algebra is the one of a maximal ideal.

**Definition 7.7 Maximal Ideal:** A proper ideal $I \in I_\Phi$ is called maximal, if $J \in I_\Phi$ and $I \subseteq J$ imply $I = J$ or $J = \Phi$.

It is well known that in a Boolean algebra a maximal ideal $I$ is also a prime ideal and vice versa, that is, if $\phi \land \psi \in I$, then either $\phi \in I$ or $\psi \in I$. Further for all $\phi \in \Phi$ either $\phi \in I$ or else $\phi^c \in I$. And also, for every ideal $I \subseteq I_\Phi$ there is a maximal ideal $I$ such that $I \subseteq I$. Finally, if $\phi \neq \psi$, then there is a maximal ideal which contains exactly one of the two elements $\phi$ or $\psi$. See (Davey & Priestley, 1990) for these results. Note that a maximal ideal represents a complete theory in the following sense:

1. the collection of information elements is consistent, because it is an ideal, that is with an element $\phi$ all implied elements $\psi \leq \phi$ belong to the collection and with $\phi$ and $\psi$ the combination $\phi \lor \psi$ belongs to the collection too.

2. it is complete, because it contains each element $\psi$ or its complement $\psi^c$, and if $\phi \land \psi$ in a theory $I$, then either $\phi$ or $\psi$ is in the theory.

In this sense, maximal ideals represent complete, consistent information.

Note that maximal ideals are atoms in the information algebra $(I_\Phi, D)$, the ideal completion of $(\Phi, D)$. According to the properties listed above $(I_\Phi, D)$ is atomic. Let $At(J)$ be the set of atoms, i.e. of maximal ideals containing ideal $J$ and $X(\Phi)$ the set of all maximal ideals of $\Phi$. Then the map $J \mapsto At(J)$ is a homomorphism of $(I_\Phi, D)$ into the set algebra $(\mathcal{P}(X(\Phi)), D)$ (see Theorem 7.6). Further, according to Section 2.2 the map $\phi \mapsto \downarrow \phi$ is an embedding of $(\Phi, D)$ into the ideal completion $(I_\Phi, D)$. So, the composed map

$$\phi \mapsto X_\phi = \{I \in X(\Phi) : \phi \in I\}$$

is a homomorphism from $(\Phi, D)$ into $(\mathcal{P}(X(\Phi)), D)$. In fact, it is an embedding: If $\phi \neq \psi$ there is a maximal ideal $I$ which contains one but not the
other of the two information elements. So $X_\phi \neq X_\psi$ and the map $\phi \mapsto X_\phi$ is one-to-one. According to the embedding of an information algebra into the subset algebra of sets, we have

$$\phi \lor \psi \mapsto X_\phi \cap X_\psi,$$

$$x(\phi) \mapsto c_x(X_\phi).$$

The operation $c_x$ is the $x$-cylindrification as defined in Section 7.2. In the present case it is defined as

$$c_x(X_\phi) = \{ J \in X(\Phi) : x(J) = x(I) \text{ for some } I \in X_\phi \}.$$

This is another representation of a Boolean information algebra by a set algebra. Based on the representation theory of Boolean algebra this representation can be described more precisely.

In Boolean representation theory the image of $\Phi$ under the map $\phi \mapsto X_\phi$ may be characterized as clopen sets in a topological space, a Stone space. In the set $X(\Phi)$ of maximal ideals of the Boolean algebra $\Phi$, a topology $\mathcal{T}$ with the sets

$$B = \{ X_\phi : \phi \in \Phi \}$$

as an open base is defined. So, the family of open sets is

$$\mathcal{T} = \{ U \subseteq X(\Phi) : U \text{ is a union of members of } \mathcal{B} \}.$$

The topological space $(X(\Phi), \mathcal{T})$ is called the dual or the prime ideal space of $\Phi$. The sets $X_\phi$ are open. Since the sets $X_{\phi'} = X_\phi^c$ are also open, the sets $X_\phi$ are also closed. The sets $X_\phi$ for $\phi \in \Phi$ are precisely the clopen subsets of $X(\Phi)$. These clopen subsets form a Boolean algebra. The space $X(\Phi)$ is compact. Further, for $I \neq J$ in $X(\Phi)$, there exists a clopen subset $X_\phi$ of $X(\Phi)$ such that $I \in X_\phi$ and $J \notin X_\phi$. That means that the topological space is totally disconnected. That is $(X(\Phi), \mathcal{T})$ is a Boolean space, namely a compact, totally disconnected topological space (Davey & Priestley, 1990). The Stone representation theorem asserts that the map

$$\phi \mapsto X_\phi = \{ I \in X(\Phi) : \phi \in I \}$$

is a Boolean algebra isomorphism of $\Phi$ onto the Boolean algebra of the clopen subsets of the dual space $(X(\Phi), \mathcal{T})$ such that

$$\phi \lor \psi \mapsto X_\phi \cap X_\psi.$$
7.4. BOOLEAN INFORMATION ALGEBRAS

\[
\phi \land \psi \mapsto X_\phi \cup X_\psi, \\
\phi^c \mapsto X_\phi^c, \\
0 \mapsto X_\Phi, \\
1 \mapsto \emptyset,
\]

This is classical Boolean representation theory (Davey & Priestley, 1990).

It remains to extend this theory to Boolean information algebras. According to the discussion above, there is a family of equivalence relations \( \equiv_x \) in \( X(\Phi) \) defined by

\[
I \equiv_x J \text{ if } x(I) = x(J)
\]

for every \( x \in D \). Here \( x \) denotes the extraction operator in the ideal extension of the information algebra \((\Phi, D)\). Note that \( x(I) = x(J) \), where \( x \) is the extraction operation in the ideal completion of the information algebra \((\Phi, D)\), holds exactly, if \( I \cap \text{Fix}_x = J \cap \text{Fix}_x \). This determines a partition of \( X(\Phi) \) and the operation of cylindrification as an operation \( c_x : \mathcal{P}(X(\Phi)) \to \mathcal{P}(X(\Phi)) \) as the saturation relative to the equivalence relation \( \equiv_x \). It is also an operator on clopen sets \( B = \{X_\phi : \phi \in \Phi\} \), that is \( c_x : B \to B \), since by the homomorphism defined above \( x(\phi) \mapsto X_{x(\phi)} = c_x(X_\phi) \). The operator \( c_x \) is an existential quantifier in the sense of Section 2.1 both on \( \mathcal{P}(X(\Phi)) \) as well as on \( B \) for every \( x \in D \), see also (Cignoli, 1991). Further the family of operators \( c_x \) for \( x \in D \) forms a commutative, idempotent semigroup of operators on \( B \), since \((B, D)\) is an information algebra, the Boolean information algebra isomorph to \((\Phi, D)\), according to the discussion above. So, we have the following representation theorem for Boolean information algebras:

**Theorem 7.10** Representation Theorem for Boolean Algebras. Let \((\Phi, D)\) be a Boolean information algebra and \((X(\Phi), T)\) the prime ideal space of the Boolean algebra \( \Phi \). Then the map

\[
\phi \mapsto X_\phi = \{I \in X(\Phi) : \phi \in I\}
\]

is an isomorphism of information algebras between \((\Phi, D)\) and the Boolean subset information algebra \((B, C(D))\), of the clopen sets \( B \) of the prime ideal space \( X(\Phi) \), and \( C(D) \) the set of all cylindrification operators for \( x \in D \), such that

\[
\phi \lor \psi \mapsto X_\phi \cap X_\psi, \\
\phi \land \psi \mapsto X_\phi \cup X_\psi, \\
\phi^c \mapsto X_\phi^c, \\
x(\phi) \mapsto c_x(X_\phi), \\
0 \mapsto X(\Phi), \\
1 \mapsto \emptyset,
\]
This calls for an extension of the concept of a Boolean space. We remind first that the family of equivalence relations \( \equiv_x \) for \( x \in D \) is monotone in the following sense: If \( x \leq y \), then \( I \equiv_y J \) implies \( I \equiv_x J \) (see Section 7.2). Further, according to Lemma 7.3 applied to the atomic information algebra \( (I \Phi, D) \), \( I \equiv_{x \land y} J \) implies that there is a maximal ideal \( H \) such that \( I \equiv_x H \) and \( J \equiv_y H \). This is used to characterize the Boolean spaces whose clopen sets form not only a Boolean algebra, but a Boolean information algebra.

**Definition 7.8** \( D \)-Boolean-Space: If \( D \) is a meet-semilattice, a Boolean space \( (X, T) \) with clopen sets \( B(X) \) is called a \( D \)-Boolean-Space, if there is a family of equivalence relations \( \equiv_x \) in \( X \) for \( x \in D \) with the associated saturation operators \( c_x \), such that

1. **Closedness**: \( \forall x \in D, c_x \) is an operator on \( B(X) \), \( c_x : B(X) \to B(X) \).

2. **Monotonicity**: \( x \leq y \) implies \( \equiv_y \subseteq \equiv_x \),

3. **Independence Property**: \( \forall u, v \in X \), with \( u \equiv_{x \land y} v \) there is a \( w \in X \) such that \( u \equiv_x w \) and \( v \equiv_y w \).

So, \( (X(\Phi), T) \), the prime ideal space of a Boolean information algebra \( (\Phi, D) \), together with the family of equivalence relations \( \equiv_x \) for \( x \in D \), is a \( D \)-Boolean-Space. According to Theorem 7.7 we note that the family of equivalence relations \( \equiv_x \) for \( x \in D \) forms a semilattice such that \( \equiv_{x \land y} = \equiv_x \lor_{\text{p}} \equiv_y \), where \( \lor_{\text{p}} \) denotes the join between equivalence relations or partitions, in other words, the equivalence relation \( \equiv_{x \land y} \) is the usual join between equivalence relations \( \equiv_x \) and \( \equiv_y \). So finally, any Boolean information \( (\Phi, D) \) is isomorph to a subset algebra over a semilattice of partitions.

To conclude this Section, we show that the clopen sets of a \( D \)-Boolean-Space form a Boolean information algebra.

**Theorem 7.11** If \( D \) is a meet-semilattice and \( (X, T) \) is a \( D \)-Boolean-Space with clopen sets \( B(X) \), and \( C(D) \) the set of all operators \( c_x \) for \( x \in D \), then \( (B(X), C(D)) \) is a Boolean information algebra.

**Proof.** The clopen sets \( B(X) \) as open and closed sets are closed under (arbitrary) intersections and unions, as well as complementation. So, \( B(X) \) is a Boolean algebra. It remains to show that \( (B(X), D) \) is an information algebra. \( B(X) \) is a join semilattice, and for all \( x \in D \), the operator \( c_x : B(X) \to B(X) \) is a quantifier on \( B(X) \).

We show that the set of operators \( C(D) \) forms a commutative, idempotent semigroup under composition. The operators form clearly a semigroup, that is, composition is associative. Let now \( x, y \in D \). Then for \( S \subseteq X \), by the monotonicity of \( \equiv_x \),

\[
c_x(c_y(S)) = \{a \in X : \exists b \in c_y(S), a \equiv_x b\}
\]
7.5. LABELED BOOLEAN ALGEBRAS

= \{ a \in X : \exists b, \exists c \in S, a \equiv_x b \equiv_y c \}
\subseteq \{ a \in X : \exists b, \exists c \in S, a \equiv_{x \land y} b \equiv_{x \land y} c \}
= c_{x \land y}(S).

Assume on the other hand \( a \in c_{x \land y}(S) \), then there is a \( c \in S \) such that \( a \equiv_{x \land y} c \). By the independence property, there is then also a \( b \) such that \( a \equiv_x b \) and \( b \equiv_y c \). But this means that \( a \in c_x(c_y(S)) \). So we have shown that \( c_x \circ c_y = c_{x \land y} \) and this is proves the desired commutativity and idempotency of operators. This concludes the proof.

In fact, in the proof above, we have shown that \( (\mathcal{P}(X), C(D)) \) itself is a Boolean subset information algebra. This concludes the discussion of the representation theory of Boolean information algebra. In the labeled case similar results hold, as will be shown in the following section.

7.5 Labeled Boolean Algebras

The results for Boolean information algebras carry easily over to labeled Boolean information algebra. If \( (\Psi, D) \) is a labeled Boolean information algebra, then for all \( x \in D \), \( \Psi_x \) is a Boolean algebra. As such it can be represented by the clopen set \( B_x \) of its prime ideal space \( X(\Psi_x) \) with

\[ X_\psi = \{ I \in X(\Psi_x) : \psi \in I \} \]

for \( \psi \in \Psi \) as its clopen sets \( B_x \). And the map \( \psi \mapsto X_\psi \) is a Boolean algebra isomorphism between the Boolean algebras \( \Psi_x \) and \( B_x \). Let

\[ B = \bigcup_{x \in D} B_x, \quad X(\Psi) = \bigcup_{x \in D} X(\Psi_x) \]

Then the map \( \psi \mapsto X_\psi \) extends in a natural way to a map from \( \Psi \) onto \( B \).

In fact, as before, the maximal ideals in \( \Psi_x \) are the atoms on \( x \) in the ideal completion \( (I_\psi, D) \) of the labeled algebra \( (\Psi, D) \), see Section 2.2. These atoms form a relational algebra \( (\mathcal{R}_{At(I_\psi)), D}) \) and \( (B, D) \) is a relational sub-algebra, which is the image of \( (\Psi, D) \) under the map \( \psi \mapsto X_\psi \)(see Section 7.2). This leads then to the following representation of a labeled Boolean information algebra:

**Theorem 7.12 Representation Theorem for Labeled Boolean Algebras.** Let \( (\Psi, D) \) be a labeled Boolean information algebra. Then the map

\[ \psi \mapsto X_\psi = \{ I \in X(\Psi_x) : \psi \in I \} \]

is an isomorphism between \( (\Psi, D) \) and the relational information algebra \( (B, D) \) of clopen sets of the prime ideal spaces \( (X(\Psi_x), T_x) \) for \( x \in D \), with
relational join as combination and relational projection \( \pi_x : \mathcal{B}_y \to \mathcal{B}_x \) for \( x \leq y \in D \) as projection, such that

\[
\begin{align*}
\phi \lor \psi & \mapsto X_\phi \bowtie X_\psi, \\
\psi^{\downarrow x} & \mapsto \pi_x(X_\psi) = \{I^{\downarrow x} : \psi \in I\}, \\
0_x & \mapsto X(\Psi_x), \\
1_x & \mapsto \emptyset,
\end{align*}
\]

As in the finite case, a labeled Boolean information algebra is isomorph to a relational algebra, this time of clopen sets of maximal ideals of \( \Psi_x \) for \( x \in D \).

As atoms of the labeled information algebra \( (I_y, D) \) the maximal ideals of \( \Psi \) form a tuple system (see Section 7.1). That is, for any \( x \leq y \in D \) there is a map \( p_{y \to x} : X(\Psi_y) \to X(\Psi_x) \) defined by \( p_{y \to x}(I) = I^{\downarrow x} \). These maps have the following properties, deduced from the properties of a tuple system:

1. **Composition:** For \( x \leq y \leq z \in D \), \( p_{y \to x} \circ p_{z \to y} = p_{z \to x} \),

2. **Independence:** For all \( x, y \in D \), if \( p_{x \to x \land y}(I) = p_{y \to x \land y}(J) \), then there is a \( H \in X(\Psi_{x \lor y}) \) such that \( p_{x \lor y \to x}(H) = I \) and \( p_{x \lor y \to y}(H) = J \),

3. **Completeness:** For all \( I \in X(\Psi_x) \) and \( x \leq y \in D \), there is a \( J \in X(\Psi_y) \) such that \( p_{y \to x}(J) = I \).

In addition, the projection \( \pi_x(X_\psi) = \{p_{y \to x}(I) : I \in X_\psi\} = X_{\psi^{\uparrow x}} \) of clopen sets in \( X(\Psi_y) \) to \( X(\Psi_x) \) are clopen sets. Finally, in the information algebra of clopen sets in \( X(\Psi) \), the vacuous extension of a clopen set of \( X_\psi \) on \( x \) to \( y \geq x \) is again a clopen set,

\[ X_\psi \bowtie E_x = p_{x \to y}^{-1}(X_\psi) = X_{\psi^{\downarrow x}}. \]

But this means that the map \( p_{x \to y} \) is a continuous map from the prime ideal space \( X(\Psi_x, \mathcal{T}_x) \) onto the space \( X(\Psi_y, \mathcal{T}_y) \), because the clopen sets \( X_\psi \) for \( \psi \in \Psi_y \) form a basis of the topology \( \mathcal{T}_y \) of the topological space \( (X(\Psi_y), \mathcal{T}_y) \).

These considerations lead to the following extension of the notion of Boolean spaces:

**Definition 7.9 D-Labeled Boolean-Space:** If \( D \) is a lattice, for all \( x \in D \), \( (X_x, \mathcal{T}_x) \) a Boolean space with clopen sets \( P(X_x) \), then the collection \( \{X_x, \mathcal{T}_x\} \) for \( x \in D \) is called a D-Labeled Boolean-Space, if there is a family of continuous maps \( p_{y \to x} : X_y \to X_x \) for all \( x \leq y \in D \) satisfying the conditions of composition, independence and completeness above and such that the map \( \pi_x : P(X_y) \to P(X_x) \), defined by

\[ \pi_x(S) = p_{y \to x}(S) = \{p_{y \to x}(a) : a \in S\} \]

maps \( P(X_y) \) into \( P(X_x) \).
7.5. LABELED BOOLEAN ALGEBRAS

The family of prime ideal spaces \((X(\Psi_x), \mathcal{T}_x)\) of a labeled Boolean information algebra form a \(D\)-labeled Boolean space.

As would be expected, \(D\)-labeled Boolean spaces induce labeled Boolean information algebras of clopen sets. Let \(D\) be a lattice and \((X_x, \mathcal{T}_x)\) for \(x \in D\) a \(D\)-labeled Boolean-space. Define

\[
P(X) = \bigcup_{x \in X} P(X_x).
\]

Next introduce the following operations:

1. Labeleling: \(d : P(X) \to D\), defined by \(d(R) = x\) if \(R \in P(X_x)\),

2. Combination: \(\otimes : P(X) \times P(X) \to P(X)\), defined for \(R, S \in P(X)\) with \(d(R) = x\) and \(d(S) = y\) by

\[
R \otimes S = R \triangleright S = \{a \in X_{x \lor y} : p_{x \lor y \rightarrow x}(a) \in R, p_{x \lor y \rightarrow y}(a) \in S\},
\]

3. Projection: \(\pi_x : P(X) \times D \to P(X)\), defined for \(x \leq d(R) = y\) by

\[
\pi_x(R) = \{p_y \rightarrow_x (a) : a \in R\}.
\]

We claim that \((P(X), D)\) endowed with these operations forms a labeled Boolean information algebra.

**Theorem 7.13** If \(D\) is a lattice and \((X_x, \mathcal{T}_x)\) for \(x \in D\) a \(D\)-labeled Boolean-space, then \((P(X), D)\) with the operations of labeling, combination and projection as defined above is a Boolean information algebra.

**Proof.** By the representation theory of Boolean algebras, for all \(x \in X\), \(P(X_x)\) is a Boolean algebra.

We show that \(R \otimes S\) belongs to \(P(X_{x \lor y})\), if \(R\) belongs to \(P(X_x)\) and \(S\) to \(P(X_y)\). Note that \(R \otimes S = p_{x \lor y \rightarrow x}^{-1}(R) \cap p_{x \lor y \rightarrow y}^{-1}(S)\). Since the maps \(p_{x \lor y \rightarrow x}\) and \(p_{x \lor y \rightarrow y}\) are continuous, the sets \(p_{x \lor y \rightarrow x}^{-1}(R)\) and \(p_{x \lor y \rightarrow y}^{-1}(S)\) are clopen and so is then their intersection \(R \otimes S\), which belongs thus to \(P(X_{x \lor y})\). Therefore, the operation of combination is well defined within \(P(X)\).

The associativity and commutativity of the combination operation are evident. The set \(X_x\) is the least element \(0_x\) for every \(x \in D\) and the empty set in \(X_x\) furnishes the top element \(1_x\) in the information ordering \(S \leq T\), if \(S \supseteq T\). The relations \(d(R \otimes S) = d(R) \lor d(S)\) and \(d(\pi_x(R)) = x\) hold by definition of these operations.

Let \(x \leq y \leq z\) and \(d(R) = z\). Then, using the composition condition above

\[
\pi_x(\pi_y(R)) = \pi_x(\{p_{z \rightarrow y}(a) : a \in R\}) = \{p_{y \rightarrow x}(b) : b \in \{p_{z \rightarrow y}(a) : a \in R\}\} = \{p_{y \rightarrow x}(p_{z \rightarrow y}(a)) : a \in R\} = \{p_{z \rightarrow x}(a) : a \in R\} = \pi_x(R).
\]
This is axiom (5) of a labeled information algebra.

Further, assume \( d(R) = x \) and \( d(S) = y \) and assume \( a \in \pi_x(R \otimes S) \). Then \( a = p_{x \vee y \to x}(c) \) for some \( c \in R \otimes S \). But then \( a = p_{x \vee y \to y}(c) \in R \) and \( p_{x \vee y \to y}(c) \in S \). From this it follows by the composition property that

\[
p_{x \to x \wedge y}(a) = p_{x \to x \wedge y}(p_{x \vee y \to x}(c)) = p_{x \vee y \to x \wedge y}(c) = p_{y \to x \wedge y}(p_{x \vee y \to y}(c)) \in \pi_{x \wedge y}(S).
\]

So, we conclude that \( a \in R \otimes \pi_{x \wedge y}(S) \).

Let now \( a \in R \otimes \pi_{x \wedge y}(S) \). This means that \( a \in R \) and \( p_{x \to x \wedge y}(a) = p_{y \to x \wedge y}(b) \) for some \( b \in S \). By the independence condition above, there is a \( c \in X_{x \vee y} \) such that \( p_{x \vee y \to x}(c) = a \) and \( p_{x \vee y \to y}(c) = b \). So \( c \in R \otimes S \) and \( p_{x \vee y \to x}(c) = a \in \pi_x(R \otimes S) \). Thus we have \( \pi_x(R \otimes S) = R \otimes \pi_{x \wedge y}(S) \). This proves axiom (6).

Next, let \( d(R) = y \) and \( x \leq y \in D \). Then,

\[
R \otimes \pi_x(R) = \{ a \in X_x : a \in R, p_{y \to x}(a) \in \pi_x(R) \} = R.
\]

If \( x \leq y \in D \), then \( p_{y \to x} \) maps \( X_y \) onto \( X_x \) by the completeness condition and this means that \( \pi_x(0_y) = 0_x \). Further, it is evident that \( 0_y \otimes 1_x = \emptyset \) hence \( 0_y \otimes 1_x = 1_y \). So, axioms (8) and (9) are satisfied.

It remains to examine condition 2 of a Boolean information algebra (see Definition 7.6). If \( d(R) = d(S) = x \leq y \), then

\[
(R \cap S) \otimes X_y = \{ a \in X_y : p_{y \to x}(a) \in R \cap S \}
= \{ a \in X_y : p_{y \to x}(a) \in R \cap \{ p_{y \to x}(a) \in X_y : a \in S \} \}.
\]

So \( (R \cap S) \otimes 0_y = (R \otimes 0_y) \cap (S \otimes 0_y) \). This concludes the proof. \( \square \)

Note that we proved in fact, that \((\mathcal{P}(X), D)\), the algebra of subsets of \( X \), is itself a labeled Boolean information, and \((P(X), D)\) is a subalgebra of it.

This completes the picture of labeled Boolean information algebras as relational algebras in topological spaces.

### 7.6 Distributive Lattice Information Algebras

There is a complete representation theory not only for Boolean algebras, but also for distributive lattices. In this Section we show how this theory extends in a natural way to distributive lattice information algebras. Here is defined what we mean by distributive lattice information algebra:

**Definition 7.10** An information algebra \((\Phi, D)\) is called a distributive lattice information algebra, if
1. The join-semilattice $\Phi$ is a distributive lattice.

2. For all $x \in D$ and for all $\phi, \psi \in \Phi$,

$$x(\phi \land \psi) = x(\phi) \land x(\psi). \quad (7.1)$$

Note that (7.1) can be derived from the properties of a Boolean information algebra (see Lemma 7.4 (6)). This condition is thus always satisfied in a Boolean information algebra, which is therefore also a distributive lattice algebra. In Section 4.5 we have also shown that (7.1) holds in any lattice induced information algebra, which therefore is again a distributive lattice information algebra (see Section 4.5). It is however so far not clear, whether it can be derived as in the Boolean case only from assuming that $\Phi$ is a distributive lattice. Note that $(\text{Fix}_x, D)$ is still a distributive information algebra.

We start by a short reminder on some elements of the representation theory of distributive lattices, (Davey & Priestley, 1990). We remind the definition of prime ideals:

**Definition 7.11** A proper ideal $I$ of a distributive lattice $\Phi$ is called prime, if for all $\phi, \psi \in \Phi$, $\phi \land \psi \in I$ implies $\phi \in I$ or $\psi \in I$.

Every maximal ideal is a prime ideal. In a Boolean algebra $\Phi$, also every prime ideal is maximal, the two notions coincide. Not so in a general distributive lattice. Let $X(\Phi)$ denote the set of prime ideals of $\Phi$. Then the map $\epsilon : \Phi \rightarrow \mathcal{P}(X(\Phi))$ defined by

$$\phi \mapsto X_\phi = \{I \in X(\Phi) : \phi \in I\}$$

is a lattice homomorphism:

$$\phi \lor \psi \mapsto X_\phi \cap X_\psi,$n

$$\phi \land \psi \mapsto X_\phi \cup X_\psi.$$

We remark that in general $X_\phi$ is defined by the condition $\phi \notin I$, so in (Davey & Priestley, 1990). Then, $\phi \lor \psi$ mapsto $X_\phi \cap X_\psi$ and $\phi \land \psi$ mapsto $X_\phi \cup X_\psi$. In ordinary lattice theory this is more natural. However in the information interpretation our view is really the natural one: Note that a prime ideal is a consistent collection of information pieces, a theory, which is complete in the sense that if $\phi$ or $\psi$ (that is $\phi \land \psi$) is affirmed, then either $\phi$ or $\psi$ must hold. Then $X_\phi$ contains all the complete theories compatible with $\phi$. And clearly, if $\phi$ is less informative than $\psi$, $\phi \leq \psi$, then more theories are compatible with $\phi$ than with $\psi$, hence $X_\psi \subseteq X_\phi$. So the information order is reversed into set-inclusion, combination of information (join in $\Phi$) becomes intersection in $X(\Phi)$. For the development of the representation theory this has otherwise no consequence.

More precisely, for distributive lattices the following theorem is well-known.
Theorem 7.14 If $\Phi$ is a lattice, then the following are equivalent:

1. The lattice $\Phi$ is distributive.

2. Given $\phi, \psi \in \Phi$ such that $\phi \not\geq \psi$, then there exists a prime ideal $I \in X(\Phi)$ such that $\phi \in I$ and $\psi \notin I$.

3. The map $\phi \mapsto X_\phi$ is an embedding of the lattice $\Phi$ into the lattice $\mathcal{P}(X(\Phi))$.

4. The lattice $\Phi$ is lattice-isomorphic to a lattice of subsets.

For a proof we refer to (Davey & Priestley, 1990).

Now we add information extraction to the picture. For this we base our development on (Cignoli, 1991). This needs some preparation. We define $x(I) = I \cap \text{Fix}_x$ for any prime ideal $I \in X(\Phi)$. This is a prime ideal in $\text{Fix}_x$. The following results are adapted from (Cignoli, 1991).

Lemma 7.5 Given $P, Q \in X(\Phi)$ such that $x(P) \subseteq x(Q)$, there is a $R \in X(\Phi)$ such that $x(R) = x(Q)$ and $P \subseteq R$.

Proof. Let $I$ be the ideal in $\Phi$ generated by $P \cap x(Q)$ and $F$ the filter in $\Phi$ generated by $\text{Fix}_x - Q$. Assume $I \cap F \neq \emptyset$ and consider $\phi \in I \cap F$. Then there must be $\eta \in \text{Fix}_x - Q$ such that $\eta = x(\eta) \leq \phi$ and a $\psi \in P$, a $\chi \in x(Q)$ so that $\phi \leq \psi \vee \chi = \psi \vee x(\chi)$. Then we obtain $\eta = x(\eta) \leq \phi \leq \psi \vee x(\chi)$ so that $\eta = x(\eta) \leq x(\psi \vee x(\psi)) = x(\psi) \vee x(\chi)$. But $\psi \in P$ implies $x(\psi) \in x(P) \subseteq x(Q)$, hence $x(\psi) \vee x(\chi) \in x(Q)$, which implies $\eta = x(\eta) \in x(Q)$ (since $x(Q)$ is an ideal in $\text{Fix}_x$). But this is a contradiction. Therefore we conclude that $I \cap F = \emptyset$.

Then by (DPI) in (Davey & Priestley, 1990) there is a prime ideal $R \in X(\Phi)$ such that $I \subseteq R$ and $R \cap F = \emptyset$. This implies $P \subseteq R$ and $R \cap (\text{Fix}_x - Q) = \emptyset$. But then also $R \cap (\text{Fix}_x - x(Q)) = R \cap \text{Fix}_x \cap x(Q)^c = \emptyset$. This implies $x(R) = R \cap \text{Fix}_x \subseteq x(Q)$. But we derive also from $I \subseteq R$ that $x(Q) \subseteq R$, hence $x(Q) = x(R)$. □

Lemma 7.6 If $P \in X(\Phi)$, $\phi \in \Phi$ and $x(\phi) \in P$, then there is a prime ideal $R \in X(\Phi)$ such that $x(R) = x(P)$ and $\phi \in R$.

Proof. If $\phi \in P$, take $R = P$. Otherwise consider the principal ideal $\downarrow \phi$ and the filter $F$ generated by $\text{Fix}_x - P$. Assume $\downarrow \phi \cap F \neq \emptyset$ and consider $\psi \in \downarrow \phi \cap F$. Then $\psi \leq \phi$ and there is a $\chi \in F$, $\chi \notin P$ such that $\chi = x(\chi) \leq \psi$. So $\chi = x(\chi) \leq \psi \leq \phi$, hence $\chi = x(\chi) \leq x(\phi)$. Since $x(\phi) \in P$ we have $\chi \in P$, which is a contradiction. So $\downarrow \phi \cap F = \emptyset$.

Again, by (DPI) in (Davey & Priestley, 1990) there is a prime ideal $Q \in X(\Phi)$ so that $\downarrow \phi \subseteq Q$ and $F \cap Q = \emptyset$. Thus, $\phi \in Q$ and $\text{Fix}_x - P = \text{Fix}_x - x(P) \subseteq \Phi - Q$. But this implies $x(Q) \subseteq x(P)$. In fact, assume
7.6. DISTRIBUTIVE LATTICE INFORMATION ALGEBRAS

\[ \psi \in x(Q), \text{ hence } \psi \not\in \Phi - Q, \text{ thus } \psi \not\in \Phi - x(P), \text{ therefore } \psi \in x(P). \]

We apply Lemma 7.5 to obtain a \( R \in X(\Phi) \) such that \( \phi \in Q \subseteq R \) and \( x(R) = x(P) \).

Based on these results we are now in a position to introduce information extraction in the lattice \( \mathcal{P}(X(\Phi)) \). For any subset \( S \) of \( X(\Phi) \) and for any \( x \in D \) define the \( x \)-cylindrification or \( x \)-saturation of \( S \) by

\[
c_x(S) = \{ P \in X(\Phi) : \exists Q \in S \text{ so that } x(P) = x(Q) \}. \tag{7.2}
\]

Here comes the main theorem for the representation of distributive lattice information algebras:

**Theorem 7.15** For all \( \phi \in \Phi \) and \( x \in D \) it holds that \( X_{x(\phi)} = c_x(X_\phi) \).

**Proof.** Consider \( P \in c_x(X_\phi) \). Then there is a \( Q \in X_\phi \) such that \( \phi \in Q \) and \( x(Q) = x(P) \). From \( x(\phi) \leq \phi \) it follows that \( x(\phi) \in x(Q) = x(P) = \text{Fix}_x \cap P \). So we see that \( x(\phi) \in P \), hence \( P \in X_{x(\phi)} \).

Conversely, let \( P \in X_{x(\phi)} \), that is, \( x(\phi) \in P \). By Lemma 7.6 there is a \( R \in X(\Phi) \) such that \( \phi \in R \) and \( x(R) = x(P) \). But this means that \( P \in c_x(X_\phi) \). This proves that \( X_{x(\phi)} = c_x(X_\phi) \). \( \square \)

Let \( U(\Phi) \) denote the image of \( \phi \) under the mapping \( \phi \mapsto X_\phi \). Since the mapping is a lattice isomorphism, \( U(\Phi) \) is a distributive lattice. But if \( (\Phi, D) \) is also an information algebra, Theorem 7.15 extends this lattice isomorphism to an information algebra isomorphism and \( (U(\Phi), D) \) becomes an information algebra with \( x \)-cylindrification as information extraction.

We return to the representation theory of distributive lattices (Davey & Priestley, 1990). The prime ideal space \( X(\Phi) \) is a topological space with basis

\[
B = \{ X_\phi \cup (X(\Phi) - X_\psi) : \phi, \psi \in \Phi \}.
\]

With the topology \( \mathcal{T}(\Phi) \) so defined, the topological space \((X(\Phi), \mathcal{T}(\Phi))\) is compact and it is ordered by inclusion. Further the clopen subsets are the finite unions of sets of the form \( X_\phi \cap (X(\Phi) - X_\psi) \) for \( \phi, \psi \in \Phi \). The clopen upsets are exactly the sets \( X_\phi \) for \( \phi \in \Phi \). So, the map \( \phi \mapsto X_\phi \) is a lattice isomorphism of the distributive lattice \( \Phi \) onto the lattice \( U(\Phi) \) of clopen upsets of the topological space \((X(\Phi), \mathcal{T}(\Phi))\). Further it holds that if \( P \not\subseteq Q \), then there is a clopen upset \( U \) such that \( P \in U \) and \( Q \not\in U \). This means that the ordered topological space \((X(\Phi), \mathcal{T}(\Phi))\) is totally order-disconnected. A compact, totally order-disconnected topological space is called a Priestley space. So \((X(\Phi), \mathcal{T}(\Phi))\) is a Priestley space.

(Cignoli, 1991) extended these results to lattices with quantifiers. Since every \( x \in D \) in a distributive lattice information algebra \((\Phi, D)\) is an existential quantifier on the distributive lattice \( \Phi \), we may apply the results in (Cignoli, 1991). Fix an \( x \in D \) and define a the relation

\[
E_x = \{ (P, Q) \in X(\Phi) \times X(\Phi) : x(P) = x(Q) \}.
\]
This is an equivalence relation in $X(\Phi)$. Associated with this equivalence relation is the cylindrification operator $c_x$ defined in (7.2). Let’s call clopen upsets $U \in \mathcal{U}(X(\Phi))$ such that $U = c_x(U)$ $x$-saturated and let $\mathcal{U}_x(X(\Phi))$ denote the set of all $x$-saturated clopen upsets. As we have seen, the information algebra $(\Phi, D)$ becomes isomorphic (as an information algebra) to the information algebra $(\mathcal{U}(\Phi), D)$, where here $D$ refers to the commutative, idempotent semigroup of operators $c_x : \mathcal{U}(\Phi) \to \mathcal{U}(\Phi)$. (Cignoli, 1991) shows further that the equivalence classes of the equivalence relation $E_x$ are closed. He derives a further result, for which we shall offer a different proof:

**Theorem 7.16** Let $P, Q \in X(\Phi)$, $x \in D$, and $(P, Q) \notin E_x$. Then there is either a $U \in \mathcal{U}_x(X(\Phi))$ such that $P \subseteq U$ and $Q \notin U$ or a $V \in \mathcal{U}_x(X(\Phi))$ such that $P \notin V$ and $Q \in V$.

We prove here first a lemma which is of interest in itself.

**Lemma 7.7** Let $\Phi$ be a distributive lattice, $\Phi'$ a sublattice and $Q \subseteq \Phi'$ a prime ideal. Then there is a prime ideal $P$ in $\Phi$ such that $Q = P \cap \Phi'$.

**Proof.** If $Q$ is a prime ideal in $\Phi'$, then $F = \Phi' - Q$ is a prime filter. Let $I$ be the ideal generated by $Q$ in $\Phi$. Similarly, let $F$ be the filter generated by $F'$ in $\Phi$. Assume that $I \cap F \neq \emptyset$. Consider a $\phi \in I \cap F$, hence a $\psi \in F$ such that $\psi \leq \phi$ and $\chi \in I'$ such that $\phi \leq \chi$. This implies $\psi \leq \phi \leq \chi$, hence $\psi, \chi \in Q \cap F' = \emptyset$. This is a contradiction, hence we have $I \cap F = \emptyset$.

Therefore, by (DPI) in (Davey & Priestley, 1990), there is a prime ideal $P$ in $\Phi$ such that $I \subseteq P$ and $P \cap F = \emptyset$. But $Q \subseteq I \cap \Phi' \subseteq P \cap \Phi'$. On the other hand, let $\phi \in P \cap \Phi'$, then $\phi \notin F$, hence $\phi \notin F'$, therefore $\phi \notin Q$. Thus, indeed, $Q = P \cap \Phi'$.

Now we continue with the proof of the theorem.

**Proof.** We note first that $Fix_x$ the set of all $\phi \in \Phi$ supported by $x$, $\phi = x(\phi)$, is a sublattice of the distributive lattice $\Phi$. Its Priestley space is $X(\Phi_x)$. Now, $(P, Q) \notin E_x$ means that $x(P) \neq x(Q)$. Hence we have either $x(P) \nsubseteq x(Q)$ or $x(Q) \nsubseteq x(P)$. In the first case the total order-disconnectedness of the Priestley space $X(Fix_x)$ means that there is a clopen upset $U \in \mathcal{U}(X(\Phi_x))$ such that $x(P) \in U$ and $x(Q) \notin U$. Let

$$U^\uparrow = \{ P \in X(\Phi) : x(P) \in U \}.$$  

Clearly $U^\uparrow$ is $x$-saturated in $X(\Phi)$. Now, $x(P) \in U$ if and only if $P \in U^\uparrow \in \mathcal{U}(X(\Phi))$. So we have $P \in U^\uparrow$ and $Q \notin U^\uparrow$. The case $x(Q) \nsubseteq x(P)$ is treated in the same way.

A consequence of this result is the following:

**Theorem 7.17** For all $x \in D$, the blocks of the equivalence relations $E_x$ are closed.
Proof. From Theorem 7.16 it follows that

\[ [P]_x = \cap \{ W \in \mathcal{U}_x(X(\Phi)) : P \in W \} \]

Since the sets \( W \) are closed, so is \([P]_x\). \( \square \)

In (Cignoli, 1991) it is proved that the closedness of the blocks of \( E_x \) implies also Theorem 7.16.

In order to extend the duality theory of distributive lattices to lattices with a quantifier, (Cignoli, 1991) defined the concept of a \( Q \)-space. We extend this concept further to obtain a duality theory of distributive lattice information algebras. For this purpose we consider a Priestley space \( X \) together with a family \( E_x \) of equivalence relations, \( x \in E \). Note that every equivalence relation gives rise to an operator \( c_x : \mathcal{P}(X) \to \mathcal{P}(X) \) in the way indicated by (7.2), namely

\[ c_x = \{ (p \in X : \exists q \in X \text{ such that } (p, q) \in E_x \} \]

We denote \( \mathcal{U}(X) \) the family of clopen upsets and by \( \mathcal{U}_x(X) \) the family of \( x \)-saturated clopen upsets, \( \mathcal{U}_x(X) = \{ U \in \mathcal{U}(X) : U = c_x(U) \} \). With these notion we define the concept of a \( EQ \)-space:

**Definition 7.12** An \( EQ \)-space is a pair \((X, E)\), where \( X \) is a Priestley space, \( E \) a set and to all \( x \in E \) an equivalence relation \( E_x \) in \( X \) is associated, such that

1. \( c_x(U) \in \mathcal{U}(X) \) for all \( U \in \mathcal{U}(X) \) and these operators on \( \mathcal{U}(X) \) form a commutative semigroup.

2. For all \( x \in E \), if \( (p, q) \notin E_x \) there is either a \( U \in \mathcal{U}_x(X) \) such that \( p \in U \) and \( q \notin U \) or a \( V \in \mathcal{U}_x(X) \) such that \( p \notin V \) and \( q \in V \).

We denote by \( C(E) \) the set of all operators \( c_x : \mathcal{U}(X) \to \mathcal{U}(X) \) for \( x \in E \) of an \( EQ \)-space. Then the clopen up-sets of the \( EQ \)-space together with this family of operators form a distributive lattice information algebra.

**Theorem 7.18** If \((X, E)\) is an \( EQ \)-space, then \((\mathcal{U}(X), C(E))\) is a distributive lattice information algebra.

Proof. By Priestley duality theory, \( \mathcal{U}(X) \) is a distributive lattice. We take intersection as combination, that is as join. Any operator \( c_x : \mathcal{U}(X) \to \mathcal{U}(X) \) is idempotent,

\[ c_x(c_x(U)) = \{ p \in X : (p, r) \in E_x, (r, q) \in E_x, q \in U \} = \{ p \in X : (p, q) \in E_x, q \in U \} = c_x(U) \]

So the operators \( C(E) \) on \( \mathcal{U}(X) \) form a commutative, idempotent semigroup. It has already been shown in (Cignoli, 1991) that each \( c_x \) is an existential
CHAPTER 7. REPRESENTATION THEORY

quantifier on \( \mathcal{U}(X) \), in fact even on \( \mathcal{P}(X) \), since, for any subsets \( S \) and \( T \) of \( X \) we have

\[
c_x(c_x(S) \cap T) = \{ p \in X : (p,q) \in E_x, q \in c_x(S) \cap T \} \\
= \{ p \in X : (p,q) \in E_x, (q,r) \in E_x, r \in S, q \in T \} \\
= \{ p \in X : (p,q) \in E_x, (q,r) \in E_x, r \in S \} \\
\cap \{ p \in X : (p,q) \in E_x, q \in T \} \\
= c_x(S) \cap c_x(T).
\]

This concludes the proof. \( \Box \)

Note that in this proof we did not use the second condition in the definition of an EQ-space. We remark further, that the family of operators \( C(\mathcal{E}) \) becomes in the usual way a meet-semilattice by defining \( c_y \leq c_x \) if \( c_y \circ c_x = c_x \). Then \( c_x \wedge c_y = c_y \circ c_x \). By abuse of language we may consider \( E \) itself as a meet semilattice where \( x \leq y \) if \( c_x \leq c_y \).

The following theorem carries over from (Cignoli, 1991) to EQ-spaces.

**Theorem 7.19** If \( (X,E) \) is an EQ-space, then the map \( \epsilon_X : X \rightarrow X(\mathcal{U}(X)) \) between \( X \) and the dual Priestley space of the distributive lattice \( \mathcal{U}(X) \) defined by

\[
\epsilon_X(p) = \{ U \in \mathcal{U}(X) : p \in U \}
\]

is a homomorphism and an order-isomorphism. Further it satisfies the condition that \( (p,q) \in E_x \) if and only if

\[
(\epsilon_X(p), \epsilon_X(q)) \in \{ (P, Q) \in \mathcal{U}(X) \times \mathcal{U}(X) : P \cap x(\mathcal{U}(X)) = Q \cap x(\mathcal{U}(X)) \}
\]

for all \( x \in E \). Here \( x(\mathcal{U}(X)) \) are the \( c_x \)-saturated sets of \( \mathcal{U}(X) \).

In order to prove this theorem condition 2 in the definition of an EQ-space is needed.

Extending the definition of Q-mappings in (Cignoli, 1991) we define EQ-mappings between two EQ-spaces \( (X,E) \) and \( (Y,F) \) as follows.

**Definition 7.13** Let \( (X,E) \) and \( (Y,F) \) be two EQ-spaces. The contravariant pair \( (f,g) \) of maps \( f : X \rightarrow Y \) and \( g : F \rightarrow E \) is called an EQ-mapping, if \( f \) is continuous and order-preserving and \( g \) a meet-homomorphism between \( F \) and \( E \) and if it satisfies for all \( y \in F \) and \( V \in \mathcal{U}(Y) \) the condition

\[
c_g(y)(f^{-1}(V)) = f^{-1}(c_g(V)).
\]

The following result is adapted from (Cignoli, 1991) to EQ-mappings.

**Lemma 7.8** For a pair of contravariant maps \( (g,f) \) between EQ-spaces \( (X,E) \) and \( (Y,F) \) where \( f : X \rightarrow Y \) is continuous and order-preserving and \( g : F \rightarrow E \) arbitrary, the following conditions are equivalent.
7.6. DISTRIBUTIVE LATTICE INFORMATION ALGEBRAS

1. \((p, q) \in E_{g(y)}\) then \((f(p), f(q)) \in F_y\).

2. \(c_{g(y)}(f^{-1}(c_y(S))) = f^{-1}(c_y(S))\) for every subset \(S\) of \(Y\).

3. \(c_{g(y)}(f^{-1}(V)) \subseteq f^{-1}(c_y(V))\) for every \(V \in \mathcal{U}(Y)\).

Proof. (1) \(\Rightarrow\) (2): Let \(S\) be a subset of \(Y\) and \(p \in c_{g(y)}(f^{-1}(c_y(S)))\). Then there is a \(q \in f^{-1}(c_y(S))\) so that \((p, q) \in E_{g(y)}\). We have

\[
f^{-1}(c_y(S)) = \{q \in X : f(q) \in c_y(S)\} = \{q \in X : [f(q)]_y \cap S \neq \emptyset\}.
\]

Now by (1) we have \((f(p), f(q)) \in F_y\) or \([f(p)]_y = [f(q)]_y\), hence \(p \in f^{-1}(c_y(S))\) and therefore \(c_{g(y)}(f^{-1}(c_y(S))) \subseteq f^{-1}(c_y(S))\). The inverse inclusion holds always. So we have equality.

(2) \(\Rightarrow\) (3): Consider a \(V \in \mathcal{U}(Y)\). Then, we have that \(f^{-1}(V) \subseteq f^{-1}(c_y(V))\), hence by (2) \(c_{g(y)}(f^{-1}(V)) \subseteq c_{g(y)}(f^{-1}(c_y(V))) = f^{-1}(c_y(V))\). This is (3).

(3) \(\Rightarrow\) (1): Consider \(p, q \in X\) such that \((f(p), f(q)) \not\in F_y\). Then we may assume that there is a \(W \in y(\mathcal{U}(Y))\) such that \(f(p) \in W\) and \(f(q) \not\in W\) (condition 2 in the definition of an EQ-space). Then \(p \in f^{-1}(W) \supseteq c_{g(y)}(f^{-1}(c_y(W)))\). Now \((p, q) \in E_{g(y)}\) implies \(q \in f^{-1}(W)\), hence \(f(q) \in W\) which is a contradiction. Therefore we conclude that \((p, q) \not\in E_{g(y)}\), which proves (1).

This leads, still according to (Cignoli, 1991), to a necessary and sufficient condition that an EQ-mapping between two EQ-spaces is an isomorphism

**Theorem 7.20** Let \((X, E)\) and \((Y, F)\) be two EQ-spaces. An EQ-mapping \((f, g)\) is an EQ-isomorphism, if and only if \(f\) is a homomorphism and an order-isomorphism, \(g\) is a meet-isomorphism and for all \(y \in F, (p, q) \in E_{g(y)}\) if and only if \((f(p), f(q)) \in F_y\).

Proof. The EQ-mapping \((f, g)\) is an EQ-isomorphism if and only if there exist the inverse mappings \(f^{-1}\) and \(g^{-1}\) and \((f^{-1}, g^{-1})\) is an EQ-mapping. This implies that \(f\) is a homomorphism and an order-isomorphism and \(g\) is a meet-isomorphism. Moreover for for all \(x \in E\) the condition

\[
c_x(f(U)) = f(c_{g^{-1}(x)}(U))
\]

is satisfied for all \(U \in \mathcal{U}(X)\). It follows from Lemma 7.8 that \((p, q) \in E_{g(y)}\) implies \((f(p), f(q)) \in F_y\) and \((u, v) \in F_{g^{-1}(x)}\) implies \((f^{-1}(u), f^{-1}(v)) \in E_x\).

Apply this to \(u = f(p), v = f(q)\) and \(y = g^{-1}(x)\) to see that \((p, q) \in E_{g(y)}\) if and only if \((f(p), f(q)) \in F_y\).

Conversely assume that \(f\) is a homomorphism and order-isomorphism, \(g\) meet-isomorphism, and that \((p, q) \in E_{g(y)}\) if and only if \((f(p), f(q)) \in F_y\).
Then $f^{-1}$ is a mapping, is also a homeomorphism and an order-isomorphism. Similarly, $g^{-1}$ is a meet-isomorphism. By Lemma 7.8,

$$c_{g(y)}(f^{-1}(V)) \subseteq f^{-1}(c_{g}(V)),$$

$$c_{g^{-1}(x)}(f(U)) \subseteq f(c_{x}(U)).$$

for every $V \in (U)(Y)$, $U \in (U)(X)$, $y \in F$, $x \in E$. Take $U = f^{-1}(V)$ and $x = g(y)$. If $V$ is a clopen upset in $Y$, then so is $U$ in $X$. Introduce this into the second identity above to obtain

$$c_{g}(V) \subseteq f(c_{g}(f^{-1}(V))).$$

This implies

$$f^{-1}(c_{g}(V)) \subseteq c_{g}(f^{-1}(V)).$$

But this implies $c_{g(y)}(f^{-1}(V)) = f^{-1}(c_{g}(V))$. In the same way we obtain $c_{g^{-1}(x)}(f(U)) = f(c_{x}(U))$. So, both $(g, g)$ as well as $(f^{-1}, g^{-1})$ are EQ-mappings, hence $(f, g)$ is an EQ-isomorphism. \hfill \Box

We remark that in the proofs of Lemma 7.8 and Theorem 7.20 we do not use that $g$ must be a meet-homomorphism, $g$ may be any mapping, in the latter case it must be of course one-to-one and onto. But this property becomes import to establish duality theory for information algebras. In fact, we may now extend duality theory of distributive lattices to distributive lattice information algebras. Consider first two distributive lattice information algebras $(\Phi, D)$ and $(\Psi, E)$ together with a homorphism $(f, g)$ from $(\Phi, D)$ into $(\Psi, E)$. Define then the mappings $D(f): X(\Phi) \to X(\Psi)$ and $D(g): C(D) \to C(E)$ by

$$D(f)(P) = f^{-1}(P), \text{ for } P \in X(\Psi), \quad D(g)(c_{x}) = c_{g(x)}, \text{ for } x \in D.$$

By standard Priestley duality theory $D(f)(V) \in X(\Phi)$ and the map $D(f)$ is continuous and order-preserving. Further, clearly, $D(g)$ is a meet-homomorphism between the semilattices $C(D)$ and $C(E)$. Further, for all $\phi \in \Phi$ we have

$$D(f)^{-1}(X_{\phi}) = X_{f(\phi)}.$$

Therefore

$$D(g)(c_{x})(D(f)^{-1}(X_{\phi})) = c_{g(x)}(X_{f(\phi)}) = X_{g(x)}(f(\phi))$$

$$= X_{f(x(\phi))} = D(f)^{-1}(X_{x(\phi)}) = D(f)^{-1}(c_{x}(X_{\phi})).$$

This shows that $(D(f), D(g))$ is an EQ-mapping between the EQ-spaces $(X(\Phi), C(D))$ and $(X(\Psi), C(E))$. This has already been shown by (Cignoli, 1991) in the context of Q-spaces.
7.7. LABELED DISTRIBUTIVE LATTICES

Similarly, let \((X, C(E))\) and \((Y, C(F))\) be two EQ-spaces and \((f, g)\) an EQ-mapping from \((Y, F)\) to \((X, E)\). Now, define the mappings \(E(f) : \mathcal{U}(X) \to \mathcal{U}(Y)\) and \(E(g) : C(E) \to C(F)\) by

\[
E(f)(V) = f^{-1}(P), \quad \text{for } V \in \mathcal{U}(Y), \quad E(g)(c_x) = c_{g(x)}, \quad \text{for } c_x \in C(E).
\]

Again, by standard Priestley duality theory, \(E(f)\) is a \(0, 1\)-lattice homomorphism from lattice \(\mathcal{U}(X)\) into lattice \(\mathcal{U}(Y)\). Also, \(E(g)\) is a meet-homomorphism from \(C(E)\) into \(C(F)\). Now, since \((f, g)\) is an EQ-map, we have by \((7.3)\) that

\[
E(f)(c_x(U)) = E(g)(c_x)(E(f)(U))
\]

for every \(V \in \mathcal{U}(X)\) and \(c_x \in C(E)\). This shows that \((E(f), E(g))\) is an information-algebra homomorphism between the distributive lattice information algebras \((\mathcal{U}(X), C(E))\) and \((\mathcal{U}(Y), C(F))\).

Let \(\eta_\Phi\) denote the information algebra isomorphism between \((\Phi, D)\) and \((\mathcal{U}(X(\Phi), C(D))\) defined by \(\eta_X(\phi, x) = (X_\phi, c_x)\). Similarly let \(\epsilon_X\) be the EQ-isomorphism between the EQ-spaces \((X, E)\) and \((X(\mathcal{U}(X)), E)\). Then, again by standard Priestley duality theory and its extension in (Cignoli, 1991) it follows that if \((f, g)\) is a homomorphism between information algebras \((\Phi, D)\) and \((\Psi, E)\), then \((f, g) \circ \eta_\Phi = \eta_\Psi \circ (ED(f), ED(g))\). And, similarly, if \((f, g)\) is an EQ-mapping between EQ-spaces \((X, E)\) and \((Y, F)\), then \((f, g) \circ \epsilon_Y = \epsilon_X \circ (DE(f), DE(g))\). This completes the discussion of duality of distributive lattice information algebra and EQ-spaces.

7.7 Labeled Distributive Lattices

It is instructive to consider also the case of labeled distributive lattice information algebras. We might consider the derived labeled version of distributive lattice information algebra. We prefer however to directly look at labeled distributive lattice information algebras and apply duality theory directly to them. Here is the definition we are going to consider.

**Definition 7.14** Labeled Distributive Lattice Information Algebra:

A labeled information algebra \((\Psi, D)\) is a labeled distributive lattice information algebra, if

1. \(\forall x \in D, \Psi_x\) is a distributive lattice, with join as combination,
2. if \(x \leq d(\phi) = d(\psi)\), then \((\phi \land \psi)^{\downarrow x} = \phi^{\downarrow x} \land \psi^{\downarrow x}\),
3. if \(x \geq d(\phi) = d(\psi)\), then \((\phi \land \psi)^{\uparrow x} = \phi^{\uparrow x} \land \psi^{\uparrow x}\),
We shall use for the combination operation both the symbol $\otimes$ as the join symbol $\lor$ at convenience. Remind that, if if $x \geq d(\phi) = d(\psi)$, then

$$(\phi \otimes \psi) \uparrow_x = (\phi \lor \psi) \uparrow_x = \phi \uparrow_x \otimes \psi \uparrow_x.$$  

We define the operators $\phi \mapsto \phi \uparrow_x$ for all $y \leq d(\phi) = x$ on $\Psi_x$.

**Lemma 7.9** The operator $\phi \mapsto \phi \uparrow_y$ is an existential quantifier on $\Psi_x$ if $y \leq x$.

**Proof.** First, we have $1_x \uparrow_y = (1_x \uparrow_y) \uparrow_x = 1_y \otimes 1_x = 1_x$. Secondly, clearly $\phi \uparrow_y \leq \phi$. Finally, using elementary properties of a labeled information algebra (see Section 4.4), we obtain

$$(\phi \uparrow_y \otimes \psi) \uparrow_y = (\phi \uparrow_y \otimes \psi) \uparrow_y \otimes 1_x = (\phi \uparrow_y \otimes 1_x) \otimes (\psi \uparrow_y \otimes 1_x) = \phi \uparrow_y \otimes \psi \uparrow_y.$$  

This completes the proof. $\Box$

So, we may apply duality theory to the distributive lattice $\Psi_x$ just as we did it in the case of an information algebra in the previous section. Consider the dual Priestley space $X(\Psi_x)$ of $\Psi_x$. Its elements are the prime ideals in $\Psi_x$. Consider the set $\Psi_x \uparrow_y = \{ \psi \in \Psi_x : \psi = \psi \uparrow_y \}$. This is still a distributive lattice, a sublattice of $\Psi_x$. If $P \in X(\Psi_x)$ is a prime ideal in $\Psi_x$, define $P \uparrow_y = P \cap \Psi_x \uparrow_y$. This is a prime ideal in the lattice $\Psi_x \uparrow_y$.

This is so far just as in a distributive lattice information algebra $(\Phi, D)$. But now we link the spaces $X(\Psi_x)$ and $X(\Psi_y)$ for $y \leq x$. Consider a prime ideal $P \in X(\Psi_x)$. We define

$$P \uparrow y = \{ \psi \uparrow y : \psi \in P \}. \quad (7.4)$$  

Note that this is the same as $P \uparrow y = \{ \psi \in \Psi_y : \psi \uparrow_x \in P \}$. Here are some results on these objects.

**Lemma 7.10** If $P \in X(\Psi_x)$, then $(P \uparrow y)^\uparrow y = P \uparrow y$.

**Proof.** Note that $\psi \uparrow_x \in P \uparrow y$ implies $\psi \uparrow y \in P$ and $\psi \uparrow x \in P$ implies $(\psi \uparrow x)^\uparrow y = \psi \uparrow y \in P \uparrow y$. So, $(P \uparrow y)^\uparrow y = \{ \psi \in \Psi_y : \psi \uparrow y \in P \uparrow y \} = \{ \psi \in \Psi_y : \psi \uparrow_x \in P \} = P \uparrow y$ as claimed. $\Box$

**Lemma 7.11** If $P \in X(\Psi_x)$, then $P \uparrow y \in X(\Psi_y)$.

**Proof.** First, $P \uparrow y$ is an ideal, since $\phi \leq \psi \in P \uparrow y$ implies $\psi \uparrow x \leq \phi \uparrow x \in P$, hence $\psi \uparrow x \in P$ and thus $\psi \in P \uparrow y$. Also, if $\psi, \phi \in P \uparrow y$, then $\psi \uparrow x, \phi \uparrow x \in P$, hence $\psi \uparrow x \lor \phi \uparrow x = (\psi \lor \phi) \uparrow x \in P$, thus $\psi \lor \phi \in P \uparrow y$. Further, if $\psi \land \phi \in P \uparrow y$, then $(\psi \land \phi) \uparrow x = \psi \uparrow x \land \phi \uparrow x \in P$ therefore, $\psi \uparrow x$ or $\phi \uparrow x$ belong to $P$, hence $\psi \in P \uparrow y$ or $\phi \in P \uparrow y$. So, the ideal $P \uparrow y$ is prime in $\Psi_y$. $\Box$

We know from Priestley duality theory that the distributive lattice $\Psi_x$ is lattice-isomorph to the lattice of clopen upsets $U(X(\Psi_x))$ containing the...
clopen upsets \( X_\psi = \{ P \in X(\Psi_x) : \psi \in P \} \). We are now going to extend the isomorphism \( \psi \mapsto X_\psi \) to an information algebra isomorphism of the whole labeled information algebra \((\Psi, D)\). Define
\[
X(\Psi) = \bigcup_{x \in D} X(\Psi_x), \quad U(\Psi(x)) = \bigcup_{x \in D} U(X(\Psi_x)).
\]
Within \( U(X(\Psi)) \) we label \( X_\psi \) by \( d(X_\psi) = x \), if \( d(\psi) = x \). Further we define a relational join between \( X_\phi \) and \( X_\psi \), if \( d(\phi) = x \) and \( d(\psi) = y \), by
\[
X_\phi \bowtie X_\psi = \{ P \in X(\Psi_{x \vee y}) : P_{\uparrow x} \in X_\phi, P_{\downarrow y} \in X_\psi \}.
\]
The following theorem shows that \( X_\phi \bowtie X_\psi \) is still a clopen upset and belongs thus to \( U(X(\Psi)) \).

**Theorem 7.21** Let \( \phi, \psi \in \Psi \). Then
\[
X_\phi \bowtie X_\psi = X_{\phi \bowtie \psi}.
\]

**Proof.** Assume first that \( P \in X_\phi \bowtie X_\psi \) and assume \( d(\phi) = x, d(\psi) = y \). Then \( P \in P_{\downarrow x} \bowtie P_{\downarrow y} \), hence \( P_{\downarrow x \vee y} \in P \). Similarly, we have also \( P_{\downarrow x \vee y} \in P \). But then \( P \in X_{\phi \bowtie \psi} \).

Conversely, consider \( P \in X_{\phi \bowtie \psi} \). Then \( (\phi \bowtie \psi)_{\downarrow x} = \phi_{\downarrow x} \bowtie \psi_{\downarrow x} \in P_{\downarrow x} \). But \( \phi \leq \phi \bowtie \psi_{\downarrow x} \), hence \( \phi \leq P_{\downarrow x} \). Similarly, we conclude that \( \psi \in P_{\downarrow y} \). But this shows that \( P \in X_{\phi \bowtie \psi} \). This concludes the proof.

Within \( X(\Psi_x) \) we have for all \( y \leq x \) an equivalence relation \( P \equiv_y Q \) if \( P^{\rightarrow y} = Q^{\rightarrow y} \). We denote the associated saturation operator by \( c_y \) (see Section 7.6). Further, we define a relational projection between \( U(X(\Psi_x)) \) and \( U(X(\Psi_y)) \) by
\[
\pi_y(X_\psi) = \{ Q_{\downarrow y} : Q \in X_\psi \}.
\]
The following two lemma prepare for the main result concerning projection.

**Lemma 7.12** For any \( x, y \in D \) with \( y \leq x \) and \( \psi \in \Psi \) with \( d(\psi) = x \), it holds that \( P \in c_y(X_\psi) \) if and only if \( P_{\downarrow y} \in \pi_y(X_\psi) \).

**Proof.** Let \( P \in c_y(X_\psi) \). Then there is a \( Q \in X_\psi \) such that \( P^{\rightarrow y} = Q^{\rightarrow y} \). But \( P^{\rightarrow y} = P_{\downarrow y} \) (Lemma 7.10). So we conclude that \( P_{\downarrow y} = Q_{\downarrow y} \), and this means that \( P_{\downarrow y} \in \pi_y(X_\psi) \).

Conversely, consider \( P_{\downarrow y} \in \pi_y(X_\psi) \). Then, \( \psi \in P \) or \( P \in X_\psi \), hence \( P \in c_y(X_\psi) \). This concludes the proof.

**Lemma 7.13** For any \( x, y \in D \) with \( y \leq x \) and \( \psi \in \Psi \) with \( d(\psi) = x \), it holds that \( P \in X_{\psi^{\rightarrow y}} \) if and only if \( P_{\downarrow y} \in X_{\psi^{\downarrow y}} \).
CHAPTER 7. REPRESENTATION THEORY

For any $\psi$ the relational projection in

Conversely, consider $P^{xy} \subseteq X_{\psi xy}$ which means that $\psi^{xy} \in P^{xy}$, which in turn means that $(\psi^{xy})^\dagger_x = \psi^{xy} \in P$, and so $P \in X_{\psi \rightarrow y}$. □

Now we are in a position to show how projection in $\Psi$ is related to relational projection in $\mathcal{U}(X(\Psi))$.

**Theorem 7.22** For any $x, y \in D$ with $y \leq x$ and $\psi \in \Psi$ with $d(\psi) = x$,

$$X_{\psi \rightarrow y} = \pi_y (X_\psi).$$

**Proof.** We show first that the mapping $P \mapsto P^{xy}$ is onto $X(\Psi_y)$. Consider a $Q \in X(\Psi_y)$. Then $R = \{ \psi^{xy} : \psi \in Q \}$ is a prime ideal in $\Psi_x^{xy}$ as is easily verified, and $R^{xy} = \{ (\psi^{xy})^\dagger_y = \psi : \psi \in Q \} = Q$. By Lemma 7.7 there is a prime ideal $P$ of $\Psi_x$ such that $P^{xy} = R$, hence by Lemma 7.10 $P^{xy} = (P^{xy})^\dagger_y = R^{xy} = Q$.

Consider now $R \subseteq X_{\psi \rightarrow y}$ for $\psi \in \Psi_x$ and $y \leq x$. Then there is a $P \in X(\Psi_x)$ so that $P^{xy} = (P^{xy})^{xy} = R$ as just shown. By Lemma 7.13 from $R = P^{xy} \subseteq X_{\psi \rightarrow y}$ it follows that $P ^{xy} \in X_{\psi \rightarrow y}$. But according to Theorem 7.15, we have $c_y(X_\psi) = X_{\psi \rightarrow y}$, hence $P \in c_y(X_\psi)$. Then finally, from Lemma 7.12, we conclude that $R = P^{xy} = \pi_y (X_\psi)$.

Conversely, assume $R \in \pi_y (X_\psi)$. This means according to the definition of relational projection that $R = P^{xy}$ for some $P \in X_\psi$. Since $X_\psi \subseteq c_y (X_\psi)$ it follows that $P \in c_y(X_\psi) = X_{\psi \rightarrow y}$. Then using Lemma 7.13 we infer that $R = P^{xy} \subseteq X_{\psi \rightarrow y}$. This proves the identity claimed in the lemma. □

The essence of Theorems 7.21 and 7.22 is that $(\Psi, D)$ and $(\mathcal{U}(X(\Psi)), D)$ are isomorphic labeled distributive lattice information algebras. This is fixed in the next theorem.

**Theorem 7.23** The labeled distributive lattice information algebras $(\Psi, D)$ and $(\mathcal{U}(X(\Psi)), D)$ are isomorphic under the map $\psi \mapsto X_\psi$, so that

$$d(\psi) = d(X_\psi),$$

$$\psi \otimes \phi \mapsto X_{\psi \otimes \phi} = X_\psi \downarrow \triangleright X_\phi,$$

$$\psi^{xy} \mapsto X_{\psi ^{xy}} = \pi_y (X_\psi).$$

For each labeled distributive lattice $\Psi_x$, $x \in D$ we have the usual Priestley duality theory. We want to extend this theory to the whole labeled information algebra $(\Psi, D)$. In the case of a labeled Boolean information algebra, the prime ideals form a tuple system since they are the atoms of the ideal completion of the algebra (see Section 7.5). In the present case of labeled distributive lattice information algebra this is no more the case, but the prime ideals form nearly a tuple system. In fact properties 1 to 3 and 5 of a tuple system (see Section 7.1) still hold, whereas property 4 is replaced by a weaker one.
Theorem 7.24 If \((\Psi, D)\) is a labeled distributive lattice information algebras, then the prime ideals in \(X(\Psi)\) satisfy the following properties, if labeling is defined by \(d(P) = x\) if \(P \in \Psi_x\) and projection \(P^{\downarrow y}\) is defined by (7.4):

1. If \(x \leq d(P)\), then \(d(P^{\downarrow y}) = x\).
2. If \(x \leq y \leq d(P)\), then \((P^{\downarrow y})^{\downarrow x} = P^{\downarrow x}\).
3. If \(d(P) = x\), then \(P^{\downarrow x} = P\).
4. If \(d(P) = x\), \(d(Q) = y\) and \(P^{\downarrow x \wedge y} = Q^{\downarrow x \wedge y}\), then for all \(\psi \in Q\) there is a \(R \in X(\Psi_{x \vee y})\) such that \(R^{\downarrow x} = P\) and \(R^{\downarrow y} \in X_\psi\).
5. If \(d(P) = x\) and \(x \leq y\), then there is a \(R \in X_y\) such that \(R^{\downarrow x} = P\).

Proof. (1) and (3) follow directly from the definition of \(P^{\downarrow y}\).
(2) By definition we have that
\[
(P^{\downarrow y})^{\downarrow x} = \{\phi^{\downarrow x} : \phi \in P^{\downarrow y}\} = \{\phi^{\downarrow x} : \phi = \psi^{\downarrow y} : \psi \in P\}
\]
\[
= \{(\psi^{\downarrow y})^{\downarrow x} : \psi \in P\} = \{\psi^{\downarrow x} : \psi \in P\} = P^{\downarrow x}.
\]

(4) Consider \(P \in X_\phi\) and \(Q \in X_\psi\) such that \(P^{\downarrow x \wedge y} = Q^{\downarrow x \wedge y}\) and \(\psi \in Q\), hence \(Q \in X_\psi\). Then \(P \in X_\phi \bowtie \pi_{x \wedge y}(X_\psi)\), since
\[
X_\phi \bowtie \pi_{x \wedge y}(X_\psi) = \{R \in X(\Psi_x) : R \in X_\phi, R^{\downarrow x \wedge y} \in \pi_{x \wedge y}(X_\psi)\}
\]
\[
= \{R \in X(\Psi_x) : R \in X_\phi, R^{\downarrow x \wedge y} = Q^{\downarrow x \wedge y}\} \text{ for some } Q \in X_\psi\}.
\]

Since \((\mathcal{U}(X(\Psi)), D)\) is a labeled information algebra with relational join as combination, we have \(\pi_x(X_\phi \bowtie X_\psi) = X_\phi \bowtie \pi_{x \wedge y}(X_\psi)\). Hence \(P \in \pi_x(X_\phi \bowtie X_\psi)\).

But
\[
\pi_x(X_\phi \bowtie X_\psi) = \{S \in X(\Psi_x) : S = R^{\downarrow x} \in X_\phi, R^{\downarrow y} \in X_\psi \text{ for some } R \in X(\Psi_{x \vee y})\}.
\]

So there is indeed a prime ideal \(R \in X(\Psi_{x \vee y})\) such that \(P = R^{\downarrow x}\) and \(R^{\downarrow y} \in X_\psi\).

(5) follows from Lemma 7.7. □

Clearly condition (4) of a tuple system implies condition (4) in the theorem above. So a tuple system is a special case of a system satisfying conditions (1) to (5) as in the theorem. As a digression we introduce now the concept of a quasi-tuple system

Definition 7.15 Quasi-Tuple System If \(D\) is a lattice, then a pair \((X, \mathcal{U})\), where \(X\) is a set, \(\mathcal{U}\) a family of subsets of \(X\), with two operations \(d : X \to D\) and \([\cdot, \cdot] : X \times D \to D\), defined for \(x \leq d(p)\) is called a quasi-tuple system over \(D\), if it satisfies the following conditions for \(U, V \in \mathcal{U}, p, q \in X\) and \(x, y \in D\):
1. if \( x \leq d(p) \), then \( d(p[x]) = x \).

2. If \( x \leq y \leq d(p) \), then \( p[y][x] = p[x] \).

3. If \( d(p) = x \), then \( p[x] = p \).

4. If \( d(p) = x \), \( d(q) = y \) and \( p[x \land y] = q[x \land y] \), then for all \( V \) such that \( q \in V \) there is a \( r \in X \) with \( d(r) = x \lor y \) such that \( r[x] = p \) and \( r[y] \in V \).

5. If \( d(p) = x \) and \( x \leq y \), then there is a \( r \in X \) with \( d(r) = y \) such that \( r[x] = p \).

6. \( \emptyset, X \in \mathcal{U} \).

7. If \( p \in \mathcal{U} \) and \( d(p) = x \), then \( \forall q \in \mathcal{U} \), \( d(q) = x \).

8. If \( p \in \mathcal{U} \) and \( d(p) = x \), \( q \in V \) and \( d(q) = y \), then the set \( \mathcal{U} \bowtie V \) defined by

\[
\mathcal{U} \bowtie V = \{ r \in X : d(r) = x \lor y, r[x] \in \mathcal{U}, r[y] \in V \}
\]

belongs to \( \mathcal{U} \).

9. If \( p \in \mathcal{U} \) and \( d(p) = x \), and \( y \leq x \), then the set \( \pi_y(U) \) defined by

\[
\pi_y(U) = \{ p[y] : p \in U \}
\]

belongs to \( \mathcal{U} \).

Note that according to Theorem 7.24 \((X(\Psi), \mathcal{U}(X(\Psi)))\) is a quasi-tuple system relative to \( D \), if \((\Psi, D)\) is a labeled distributive lattice information algebra. We note next, that any quasi-tuple system gives rise to a labeled information algebra.

**Theorem 7.25** Let \((X, \mathcal{U})\) be a quasi-tuple system relative to a lattice \( D \). Then \((\mathcal{U}, D)\) is a labeled information algebra with relational join \( \mathcal{U} \bowtie V \) for \( U, V \in \mathcal{U} \) as combination and relational projection \( \pi_y(U) \) for \( U \in \mathcal{U} \) and \( y \in D \) as projection.

**Proof.** We verify the axioms of a labeled information algebra (see Section 4.4).

Define for \( U \in \mathcal{U} \), \( d(U) = x \) if for a \( p \in U \) \( d(p) = x \). This defines the labeling operation. Define further for \( x \in D \), \( 0_x = \pi_x(X) \) and \( 1_x = \pi_x(\emptyset) \).

The clearly \( 0_x \bowtie U = U \) and \( 1_x \bowtie U = 1_x \) for any \( U \in \mathcal{U} \) with \( d(U) = x \). Associativity and commutativity of the relational join operation are evident. Also \( d(U \bowtie V) = d(U) \cup d(V) \) and \( d(\pi_x(U)) = x \) follows directly from the definitions of these operations.
Consider then \( x \leq y \leq d(U) \) for some \( U \in \mathcal{U} \). Then, using (2) in the definition of a quasi-tuple system,

\[
\pi_x(\pi_y(U)) = \pi_x(\{p[y] : p \in U\}) = \{q[x] : q = p[y], p \in P\} \\
= \{p[y][x] : p \in P\} = \pi_x(U).
\]

In order to show that for \( U, V \in \mathcal{U} \) with \( d(U) = x \) and \( d(V) = y \), we have \( \pi_x(U \Join V) = U \Join \pi_x \wedge y \) consider first a \( p \in \pi_x(U \Join V) \). This means that there is a \( s \in X \) such that \( d(s) = x \lor y \) and \( s[x] = p \in U, s[y] \in V \). Define \( q = r[y] \). Then \( p[x \land y] = q[x \land y] \) and \( q \in V \). But this shows that \( p \in U \Join \pi_x \wedge y (V) \). Conversely, if \( p \in U \Join \pi_x \wedge y (V) \), then \( p \in U \) and \( p[x \land y] = q[x \land y] \) for some \( q \in V \). By (4) in the definition of a quasi-tuple system, there is then a \( r \in X \) with \( d(r) = x \lor y \) such that \( r[x] = p \in U \) and \( r[y] \in V \). This means that \( p \in \pi_x(U \Join V) \). So the identity claimed above holds indeed.

If \( U \in \mathcal{U} \) and \( y \leq d(U) \), then clearly \( U \Join \pi_y(U)U \). Also if \( x \leq y \), then \( \pi_x(0_y) = \pi_x(\pi_y(X) = \pi_x(X) = 0_x \). And finally \( 0_y \Join 1_x = \pi_y(X) \Join \pi_x(\emptyset) = \pi_x(\emptyset) = 1_x \). This completes the proof. \( \Box \)

This motivates the following definition a labeled EQ-space.

**Definition 7.16 Labeled EQ-Space:** A system \((X, D)\) is called a labeled EQ-space, if

1. \( D \) is a lattice.
2. \( X = \cup_{x \in D} X_x \) and \( \forall x \in D, X_x \) is a Priestley space.
3. \((X, \mathcal{U}(X))\), where \( \mathcal{U}(X) = \cup_{x \in D} \mathcal{U}(X_x) \), is a quasi-tuple system over \( D \) with labeling defined by \( d(p) = x, \) if \( p \in X_x \).

As a corollary of Theorem 7.25 and Priestley duality theory for distributive lattices, it follows that \((\mathcal{U}(X), D)\) is a labeled distributive lattice information algebra.

**Corollary 7.1** If \((X, D)\) is a labeled EQ-space, then \((\mathcal{U}(X), D)\) is a labeled distributive lattice information algebra.

**Proof.** By Theorem 7.25 \((\mathcal{U}(X), D)\) is a labeled information algebra. By Priestley duality theory any \((\mathcal{U}(X_x))\) is a distributive lattice with intersection as join and union as meet. It remains to show that for \( U, V \in \mathcal{U}(X_y) \) and \( x \leq y \), \( \pi_x(U \cup V) = \pi_x(U) \cup \pi_x(V) \) and for \( y \geq x \), \((U \cup V)^\uparrow y = U^\uparrow y \cup V^\uparrow y \). The first identity is evident. To verify the second note that \( U^\uparrow y = U \Join 0_y = U \Join \pi_y(X) \). The second identity becomes evident too. \( \Box \)

The development of the whole duality theory between labeled distributive information algebras and labeled EQ-spaces along similar lines as in Section 7.6 remains to be done. Also missing is the exploration of the relations between distributive lattice information algebras and their labeled versions as well as between the dual EQ-spaces and their labeled versions.
7.8 Finite Distributive Lattices
Chapter 8

Topology

8.1 Properties and Topology

A property of information is a feature shared by several pieces of information. Formally it is thus simply a subset of the set Φ of all pieces of information. For instance being a finite piece of information is such a property, or being more informative than some fixed element φ is another property. However, not every subset can be considered to define a property. Informally, there must be some procedure which detects a given property \( P \) of a piece of information. Such a procedure detects a property \( P \) of an element \( \phi \) with some kind of finite resources, for example finite time. And only features which can be verified with such finite resources are considered as properties. Then two important closure properties of properties may be required (Smyth, 1992):

1. **Finite Conjunction:** Suppose \( P_1, \ldots, P_n \) are properties. Then if each property \( P_i \) can be verified with a finite amount of resources, then so can the conjunction \( P_1 \land P_2 \) and so on, up to \( P_n \). Hence the intersection \( P_1 \cap \ldots \cap P_n \) is also property.

2. **Arbitrary Disjunction:** Suppose \( \mathbf{P} \) be any collection of properties. Any property \( P \in \mathbf{P} \) can be verified with finite resources an a piece of information \( \phi \). But then it is verified that a property of the collection \( \mathbf{P} \) is satisfied. This means that the union \( \bigcup \mathbf{P} \) is a property.

These are exactly the properties of open sets of a topology. In other words the properties of a set of pieces of information \( \Phi \) define a topology \( \mathcal{T} \) in \( \Phi \). Somehow, a property \( P \in \mathcal{T} \) is a meta-information, an information about pieces of information. Not too surprisingly we shall show in this section that reasonable topologies in information algebras form themselves information algebras. This is the main subject of the present chapter.

As an example, we illustrate the concept of *finitely observable properties* with respect to *strings*. We have seen that strings form an information
algebra, see Section 3.1. Let $\Sigma$ be a finite alphabet and $\Sigma^{**} = \Sigma^* \cup \Sigma^\omega \cup \{z\}$ the set of finite and infinite strings over this alphabet. A finitely observable property of a string from $\Sigma^{**}$ is something that can be verified on a finite prefix of the string. So, more formally, a finitely observable property $P$ of a string is a subset of $\Sigma^{**}$ such that, if a string $s$ belongs to $P$, then there is some finite initial prefix $s^{\downarrow n}$ of $n$ symbols such that any extension of $s^{\uparrow n}$ belongs to $P$. This says exactly, that if property $P$ holds on some output string $s$, then some finite initial segment of this output is sufficient to verify the property. Note that this does not mean that the property is decidable. If a string $s$ does not belong to $P$, then this can in general not be verified on a finite initial segment of $P$. For instance, the property that there are at least ten occurrences of the symbol 0 in a string with the alphabet $\{0, 1\}$ is finitely observable, since if a string $s$ has this property it can be verified on the finite initial segment up to the occurrence of the tenth symbol 0. However, if $s$ does not have this property, it can not be verified on any finite prefix of $s$.

An example of a not finitely observable property is the property of a string having only a finite number of occurrences of the symbol 0.

If $s$ is a finite string of length $n$, then let $s^{\downarrow \omega} = \{r \in \Sigma^{**} : r^{\uparrow n} = s\}$ be the set of strings (finite or infinite) having $s$ as a prefix. The set $s^{\downarrow \omega}$ represents a finitely observable property, since it can be verified on the finite string $s$. Further, if the subset $P$ of $\Sigma^{**}$ represents a finitely observable property, then any element of $P$ must belong to some $s^{\downarrow \omega}$, that is have a finite prefix which allows to verify the property. This means that

$$P = \bigcup \{s^{\downarrow \omega} : s \in \Sigma^*, s \leq r \text{ for some } r \in P\}.$$

If $P_1, \ldots, P_n$ are observable properties, then

$$\bigcap_{i=1}^n P_i = \bigcap_{i=1}^n \bigcup \{s^{\downarrow \omega} : s \in \Sigma^*, s \leq r \text{ for some } r \in P_i\}$$

$$= \bigcup \{s^{\downarrow \omega} : s \in \Sigma^*, s \leq r \text{ for some } r \in \bigcap_{i=1}^n P_i\}.$$

Similarly, if $P_i, i \in I$ is an arbitrary collection of finitely observable properties, then

$$\bigcup_{i \in I} P_i = \bigcup_{i \in I} \bigcup \{s^{\downarrow \omega} : s \in \Sigma^*, s \leq r \text{ for some } r \in P_i\}$$

$$= \bigcup \{s^{\downarrow \omega} : s \in \Sigma^*, s \leq r \text{ for some } r \in \bigcup_{i \in I} P_i\}.$$

This shows that the family of finitely observable properties satisfy the conditions of finite conjunction and arbitrary disjunction, hence define a topology $T$ on $\Sigma^{**}$. This is an example of a more general topology associated with a compact information algebra, see Section 8.3.
8.2 Alexandrov Topology

In this section we consider an information algebra \((\Phi, D)\). The set \(\Phi\) is partially ordered and it is well known, that with such a set one can associate the topology \(T\) of upwards closed sets (upsets). A set \(U\) is an upset, if \(\phi \in U\) and \(\phi \leq \psi\) implies \(\psi \in U\). It is evident that that the family of upsets is closed under finite intersections and arbitrary unions. The family \(B\) of sets
\[\uparrow \phi = \{ \psi \in \Phi : \phi \leq \psi \}\]
is a base of the topology \(T\), since for any \(U \in T\),
\[U = \bigcup_{\phi \in U} \uparrow \phi.\]
The topology \(T\) is called Alexandrov topology (Smyth, 1992). If an information \(\phi \) has property \(P\), then all more informative elements \(\psi \geq \phi\) share this property.

We claim that the open sets as well as the base sets of this topology form themselves an information algebra. We first examine the base sets. In \(B\) the elements \(\uparrow \phi\) are ordered by inclusion, \(\uparrow \psi \subseteq \uparrow \phi\) if \(\uparrow \phi \subseteq \uparrow \psi\). Then
\[\uparrow \phi \vee \uparrow \psi = \uparrow (\phi \vee \psi).\]

Further, for \(x \in D\) we define
\[x(\uparrow \phi) = \bigcup_{\phi \in U} \uparrow x(\phi).\]

This gives us an information algebra, in fact an algebra isomorph to \((\Phi, D)\).

**Theorem 8.1** If \((\Phi, D)\) is an information algebra, then \((B, D)\) is also an information algebra, isomorph to \((\Phi, D)\).

**Proof.** Clearly \(B\) is a semilattice, isomorph to \(\Phi\) under the mapping \(\phi \mapsto \uparrow \phi\). The elements \(x \in D\) define operators \(x : B \to B\) as defined above which form a commutative, idempotent semigroup. The map \(\phi \mapsto \uparrow \phi\) extends to information algebra homomorphism, is onto \(B\) and one-to-one.

In \(T\), the inclusion order defines also a semilattice by \(U \vee V = U \cap V\). For any \(x \in D, U \in T\) we define
\[x(U) = \bigcup_{\phi \in U} \uparrow x(\phi).\]

Clearly \(x(U)\) is an open set and hence \(x\) an operator \(x : T \to T\). This again defines an information algebra.

**Theorem 8.2** If \((\Phi, D)\) is an information algebra, then \((T, D)\) is also an information algebra, having \((B, D)\) as a subalgebra.
Proof. It remains only to verify that the operators in $D$ form a commutative semigroup of existential quantifiers. Let $U \in \mathcal{T}$. Then

$$x(U) = \bigcup_{\phi \in U} \uparrow x(\phi) = \bigcup_{\phi \in x(U)} \uparrow \phi.$$ 

Thus,

$$(x \circ y)(U) = y(x(U)) = \bigcup_{\phi \in x(U)} \uparrow y(\phi)$$

$$= \bigcup_{\phi \in U} \uparrow y(\phi) = \bigcup_{\phi \in U} \uparrow x(y(\phi)) = (y \circ x)(U).$$

So $D$ is a commutative semigroup of operators $x : \mathcal{T} \to \mathcal{T}$. It is by the same analysis also idempotent, since the operators $x : \Phi \to \Phi$ are idempotent.

The top element in the semilattice $\mathcal{T}$ is $\Phi$. Now $x(\Phi) = \uparrow \Phi = \Phi$. Further, since $\uparrow \phi \subseteq \uparrow x(\phi)$. It follows also that $U \subseteq x(U)$. Finally, we have

$$x(x(U) \cap V) = x \left( \left( \bigcup_{\phi \in U} \uparrow x(\phi) \right) \cap \left( \bigcup_{\psi \in V} \uparrow \psi \right) \right)$$

$$= x \left( \bigcup_{\phi \in U; \psi \in V} \uparrow \left( x(\phi) \lor \psi \right) \right)$$

$$= \bigcup_{\phi \in U; \psi \in V} \uparrow x(\phi) \lor \psi).$$

On the other hand

$$x(U) \cap x(V) = \left( \bigcup_{\phi \in U} \uparrow x(\phi) \right) \cap \left( \bigcup_{\psi \in V} \uparrow x(\psi) \right)$$

$$= \bigcup_{\phi \in U; \psi \in V} \uparrow \left( x(\phi) \lor x(\psi) \right).$$

This shows that $x(x(U) \cap V) = x(U) \cap x(V)$ and this shows that $x$ is an existential quantifier on $\mathcal{T}$ and this concludes the proof.  

We remark that the information algebra $(\mathcal{T}, D)$ is a distributive lattice information algebra. Thus an information algebra $(\Phi, D)$ is embedded into a distributive lattice information algebra.

8.3 Scott Topology

In this section a topology of a continuous information algebra $(\Phi, D)$ will be considered. We define a topology $\mathcal{S}$ whose open sets are subsets $U \in \Phi$ such that
8.3. Scott Topology

1. $U$ is an upset,

2. If $X$ is directed, and $\forall X \in U$, then there is a $\psi \in X \cap U$.

We know that $\Phi$ is a continuous lattice and for such lattices this topology is defined in (Scott, 1971). Therefore we call it the Scott Topology. From the second condition, if $B$ is a basis, it follows that if $\phi \in U$, $U$ open, then there is a $\psi \in U \cap B$ such that $\psi \ll \phi$. Note that in particular, if $(\Phi, D)$ is compact with finite elements $\Phi_f$, hence the lattice $\Phi$ algebraic, then the definition above is equivalent to

1. $U$ is an upset,

2. If $\phi \in U$, then there is a $\psi \leq \phi$ such that $\psi \in \Phi_f \cap U$.

This concurs with the idea of a finitely observable property. If a piece of information $\phi$ has a property $U$, then there is finite information $\psi$, less informative than $\phi$, which already has this property, which means that the property is “observable” on a finite piece of information. Further the set $U_\psi = \{ \phi \in \Phi : \phi \not\leq \psi \}$ is open. If $\phi \neq \psi$, then either $\phi \not\leq \psi$ or $\psi \not\leq \phi$. In the first case, $\phi \in U_\psi$ and $\psi \not\in U_\psi$. in the second case, $\phi \not\in U_\psi$ and $\psi \in U_\psi$. This shows that for any pair of elements $\phi \neq \psi \in \Phi$ there is an open set $U$ such that either $\phi \in U$ and $\psi \not\in U$ or $\psi \in U$ and $\phi \not\in U$. That is, the topological space $(\Phi, S)$ is a $T_0$-space. Continuous lattices and $T_0$-spaces are closely related as has been shown in (Scott, 1971).

The sets $\uparrow \phi = \{ \psi : \phi \ll \psi \}$ form a base $\uparrow \Phi$ for the topology $S$ (Scott, 1971). In fact, for any open set $U \in S$ it holds that

$$U = \bigcup_{\phi \in U} \uparrow \phi.$$  

As before, this base can be considered as an information algebra. With inclusion as partial order, the join is

$$\uparrow \phi \vee \uparrow \psi = \uparrow \phi \cap \uparrow \psi.$$  

Thus $\uparrow \Phi$ becomes a join-semilattice. For any $x \in D$ we define the extraction operator

$$x(\uparrow \phi) = \uparrow x(\phi).$$  

This defines an information algebra isomorph to $(\Phi, D)$.

**Theorem 8.3** If $(\Phi, D)$ is a continuous information algebra, then $(\uparrow \Phi, D)$ is also an information algebra, isomorph to $(\Phi, D)$.
Proof. We have seen that $\| \Phi$ is a semilattice. By Lemma 6.6 (1) we obtain that $\| \phi \cap \| \psi \leq \{ \chi : \phi \lor \psi \ll \chi \}$. On the other hand, $\phi \lor \psi \ll \chi$ implies $\phi, \psi \ll \chi$, hence $\| \phi \cap \| \psi = \| (\phi \lor \psi)$. Further, $\| \mathbf{0} = \Phi$, and $\| \mathbf{1} = \emptyset$. So $\| \Phi$ is semilattice-isomorph to $\Phi$ under the mapping $\phi \mapsto \| \phi$. The elements $x \in D$ represent operators $x : \| \Phi \rightarrow \| \Phi$ as defined above, which form a commutative, idempotent semigroup. This follows directly from the definition of the operators and their properties in $(\Phi, D)$. It remains to show that the operators are existential quantifiers on $\| \Phi$. By definition $x((\mathbf{1}) = \| x(1) = \| \mathbf{1} = \emptyset$, which is property (a) of an existential quantifier. Then clearly $x((\phi) = \| x(\phi) \supseteq \| \phi$, since $x(\phi) \leq \phi$. This is property (b). Finally,

$$x(\| x(\phi) \cap \| \psi) = x(\| x(\phi) \cap \| \psi)$$

$$= x((x(\phi) \lor \psi))$$

$$= \| x(x(\phi) \lor \psi) = \| (x(\phi) \lor x(\psi))$$

$$= \| (x(\phi)) \cap \| (x(\psi)) = x(\| \phi) \cap x(\| \psi).$$

So, property (c) holds too. Hence $(\| \Phi, D)$ is an information algebra, and the map $\phi \mapsto \| \phi$ is an information algebra homomorphism, is onto $\| \Phi$ and one-to-one. □

Now, we show that the open sets $U$ of $\mathcal{S}$ form also an information algebra $(\mathcal{S}, D)$. The sets of a topological space are ordered under inclusion and form a complete lattice under this order. So, in particular they form a join-semilattice, if we define

$$U \lor V = U \cap V.$$

Further, for any $x \in D$ and $U \in \mathcal{S}$, we define

$$x(U) = \bigcup_{\phi \in U} \| x(\phi).$$

The set $x(U)$ is obviously open as an union of base-sets. We need the following lemma.

Lemma 8.1 For any set $I$, If $U = \bigcup_{i \in I} \| \phi_i$, then $x(U) = \bigcup_{i \in I} \| x(\phi_i).$

Proof. First, we show monotonicity of the operator $x$: If $V, U$ open and $V \subseteq U$, then $x(V) \subseteq x(U)$. In fact, $V = \bigcup_{\psi \in V} \| \psi$ and $U = \bigcup_{\psi \in U} \| \psi$. So, if $\psi \in V$, then $x(\psi) \subseteq \bigcup_{\psi \in U} \| x(\psi) = x(U)$. Therefore we conclude that $x(V) = \bigcup_{\psi \in V} \| x(\psi) \subseteq x(U)$. So, in particular, $\| \phi_i \subseteq U$ implies $x(\| \phi_i) = \| x(\phi_i) \subseteq x(U)$, hence $\bigcup_{i \in I} \| x(\phi_i) \subseteq x(U)$.

Conversely, consider $\eta \in x(U)$. Then there is a $\psi \in U$ so that $x(\psi) \ll \eta$. But $\psi \in U$ implies that there is an $i \in I$ such that $\phi_i \ll \psi$. This in turn implies $x(\phi_i) \leq x(\psi)$, therefore $x(\phi_i) \ll \eta$, that is, $\eta \in \| x(\phi_i)$. So we see that $\eta \in \bigcup_{i \in I} \| x(\phi_i)$. This shows that $x(U) = \bigcup_{i \in I} \| x(\phi_i)$. □

Now, we are in a position to prove the following theorem.
8.3. SCOTT TOPOLOGY

Theorem 8.4 If \((\Phi, D)\) is a continuous information algebra, then \((S, D)\) is also an information algebra, having \((|\Phi, D)\) as a subalgebra.

Proof. We have already noted that \(S\) is a join-semilattice under inclusion, with intersection as join (corresponding to the information order). It has \(\emptyset\) as top and \(\Phi\) as bottom element.

The elements \(x \in D\) define operators \(x : S \rightarrow S\) also as noted above. We show that these operators form a commutative, idempotent semigroup. Consider \(x, y \in D\) and \(U \in S\). Then,

\[
x(y(U)) = x\left( \bigcup_{\psi \in U} |y(\psi)| \cup \bigcup_{\psi \in U} |x(y(\psi))| \right)
\]

by Lemma 8.1. Now, commutativity holds for \(x\) and \(y\) as operators of \(\Phi\). This shows that it holds also as operators on \(S\). In the same way, we see that \(x(U) = x(U)\), since idempotency holds for \(x\) as an operator of \(\Phi\).

It remains to show that \(x\) as an operator of \(S\) is an existential quantifier. Clearly we have \(x(\emptyset) = \emptyset\). Further from \(|\psi| \subseteq |x(\psi)|\) it follows that \(U \subseteq x(U)\), so, in the information order, \(x(U) \leq U\). Finally, again using Lemma 8.1,

\[
x(U \cap V) = x\left( \bigcup_{\phi \in U} |x(\phi)| \cap \bigcup_{\psi \in V} |x(\psi)| \right) = x\left( \bigcup_{\phi \in U, \psi \in V} |x(\phi) \lor x(\psi)| \right)
\]

This concludes the proof. \(\square\)

In addition of being an information algebra, \((S, D)\) is a distributive lattice information algebra, that is for \(U, V \in S\),

\[
x(U \cup V) = \left( \bigcup_{\phi \in U \cup V} |\phi| \right) = \left( \bigcup_{\phi \in U \cup V} |x(\phi)| \right) = x(U) \cup x(V),
\]

where the set union corresponds to meet in the information order, so that \(x(U \wedge V) = x(U) \wedge x(V)\).

To conclude this section, let’s look at continuous maps \(f\) between two \(D\)-continuous information algebras \((\Phi, D)\) and \((\Psi, E)\). In Section 6.7 we
have seen that these maps form themselves a $D \times E$-continuous information algebra $([\Phi \to \Psi]_c, D \times E)$. Select $x \in D$ and $y \in E$. Then $\phi \equiv_x \psi$ if $x(\phi) = x(\psi)$ is an equivalence relation in $\Phi$, $\phi \equiv_y \psi$ if $y(\phi) = y(\psi)$ is an equivalence relation in $\Psi$, and $f \equiv_{x,y} g$ if $(x,y)(f) = (x,y)(g)$ is an equivalence relation in $[\Phi \to \Psi]_c$. Suppose that $f \equiv_{x,y} g$ and $\phi \equiv_x \psi$. Then $y(f(x(\phi))) = y(g(x(\psi)))$, hence $f(x(\phi)) \equiv_y g(x(\psi))$ that is $(f \circ x)(\phi) \equiv_y (g \circ x)(\psi)$. Conversely, suppose $\phi \equiv_x \psi$ implies $(f \circ x)(\phi) \equiv_y (g \circ x)(\psi)$. Then, since for all $\phi \in \Phi$ we have $\phi \equiv_x \phi$, it follows $y(f(x(\phi))) = y(g(x(\phi)))$, hence $(x,y)(f) = (x,y)(g)$ or $f \equiv_{x,y} g$. This means that $f \circ x$ and $g \circ x$ are equivariant maps relative to the equivalence relations $\phi \equiv_x \psi$ and $\phi \equiv_y \psi$ (Scott, 1998). This puts continuous information algebras into the context of equilogical spaces, but it is so far not clear, whether this has interesting consequences for information algebras.

topology is a lattice algebra... (see LN for compact algebras)

Skierpinsky inf. algebra and embedding of continuous algebras ???

inf. extraction and projections (are extractions projections, is a $T_0$ space together with all its projections an (D-continuous) inf. algebra?)

### 8.4 Topology of Labeled Algebras

Generally introduce $T_0$ spaces and show that specialisation order coincides with information order in Scott- and Alexandrov topologies.

Open question: What is the topological characterization of $T_0$-spaces whose specialisation order is a semi-lattice (i.e. an information algebra)?
Chapter 9

Uncertain Information

9.1 Introduction

In practice it can not be excluded that contradictory information is asserted. Then at least one of these assertions must be wrong. This immediately leads to the idea that information may be uncertain, at least in the sense that its assertion may be wrong. For instance, if the source of an information is a witness, an expert or a sensor, there is always the possibility that the witness lies, the expert errs or that the sensor is faulty. More generally, the truth of a piece of information may depend on certain assumptions whose validity is uncertain. Turned the other way round: Assuming certain assumptions out of a set of possible assumptions, certain pieces of information may be asserted. The uncertainty of the information stems from the uncertainty about which assumption is valid. Also different assumptions may have different likelihood or probabilities to be valid. Viewed from this angle, uncertain information is represented by a map from a probability space into an information algebra.

Given such a map, for any piece of information in the information algebra, or more generally each consistent system of information in its ideal completion, the assumptions supporting the information can be determined: These are all the assumptions whose validity entails the information. The probability of the assumptions supporting a piece of information measures the degree of support of it. Here enters the question of the measurability of the support. To overcome the restrictions imposed by measurability considerations, allocations of probability in the probability algebra associated with the probability space of assumptions can be considered (Kappos, 1969; Shafer, 1973).

Maps representing uncertain information inherit the structure of an information algebra from their range. Uncertain information thus still is information. In many cases, finite uncertain information is in a natural way to be defined, which turns these algebras of uncertain information into compact
information algebras.

The present concept of uncertain information has its roots in the theory of hints (Kohlas & Monney, 1995) which in turn is based on Dempster’s multivalued mappings (Dempster, 1967a). However, whereas Dempster derives probability bounds form these multivalued mappings, the semantics of the theory of hints is in the spirit of assumption-based reasoning as sketched above. As seen from the point of view of information algebra, hints are mappings into subset algebra, in particular, a multivalued set algebra (see Sections 3.3 and 3.4). The theory can also be given a logical flavor. It may for instance be combined with propositional logic (Haenni et al., 2000; Kohlas, 2003a). Since this approach combines logic for deduction of arguments with probability to evaluate likelihood of arguments, we speak also of probabilistic argumentation systems. This corresponds to mappings into the information algebra associated with propositional logic (see Section 4.3). A more abstract presentation of this point of view is given in (Kohlas, 2003b).

Dempster’s approach to multivalued mappings was given by Shafer a more epistemological flavor (Shafer, 1976). The primary object in this view is the belief function which corresponds to our degree of support and leads to an allocation of probability as hinted above (Shafer, 1973). Therefore, in the spirit of Shafer, we study allocations of belief and show that they too lead to information algebras. In particular, we study how these allocations of probabilities relate to the mappings representing uncertain information. Allocations of probability generate a mathematical object called here support function and a related one called plausibility function. These objects correspond to distribution functions of random variables. It is well known in probability theory that any distribution function is induced by a random variable. In the same vain, we shall show that any such object can be generated by a mapping representing uncertain information. Support functions on information algebras will be studied in Chapter 10.

In the next section, the simplest way to represent uncertain information will be presented as a model for more involved models, to be discussed in subsequent sections.

9.2 Simple Random Variables

What do we mean by uncertain information? It is information we are not certain that it is valid, that it is correct. We present in this section a first approach to model this situation. In our context, information is represented as always by elements of an information algebra \((\Phi, \mathcal{D})\). Up to now, stating or asserting an information means to select an element \(\phi \in \Phi\). Then the principal ideal \(\downarrow \phi\) represents the consistent corpus of information pieces asserted. More generally any ideal in \(\Phi\) is such a consistent corpus. Now, we may admit, that such an information can only be asserted if certain
assumptions hold; assumptions we are not sure whether they are valid. If not, may be we must admit that we have no information. More generally, we may admit several, mutually exclusive assumptions, each giving rise to a possibly different corpus of information. We may furthermore assume different likelihoods or probabilities for these assumptions to be valid. This gives then rise to a structure of uncertain information, which we shall formalize in this chapter. This chapter is an adaption and extension of material in (Kohlas, 2003a)

Let $\Omega$ be a set whose elements represent assumptions. In applications, $\Omega$ often will be a finite set. But we drop this requirement for the sake of generality. In order to introduce probability, we assume $(\Omega, A, P)$ to be a probability space with $A$ a $\sigma$-algebra of subsets of $\Omega$ and $P$ a probability measure on $A$. Uncertain information will be represented by a map from $\Omega$ to $\Phi$, where $\Phi$ is an information algebra. In order to simplify, and for considerations of measurability, which will be dropped later, we restrict however in a first step the maps to be considered.

Let $B = \{B_1, \ldots, B_n\}$ be any finite partition of $\Omega$, whose blocks $B_i$ belong all to $A$. A mapping $\Delta : \Omega \to \Phi$, such that $\Delta(\omega)$ is constant for all $\omega$ of a block $B_i$,

$$\Delta(\omega) = \phi_i, \text{ for all } i \in B_i,$$

is called a simple random variable in $\Phi$. Denote the family of all simple random variables by $R_s$. These maps inherit the operations of the information algebra:

1. **Combination:** Let $\Delta_1$ and $\Delta_2$ be simple random variables in $(\Phi, D)$. Then $\Delta_1 \lor \Delta_2$ is defined pointwise by

$$ (\Delta_1 \lor \Delta_2)(\omega) = \Delta_1(\omega) \lor \Delta_2(\omega), $$

where on the right join is taken in $\Phi$.

2. **Extraction:** Let $\Delta$ be a simple random variable in $(\Phi, D)$. Then define $x(\Delta)$ by

$$ (x(\Delta))(\omega) = x(\Delta(\omega)), $$

where on the right extraction takes place in $\Phi$.

We have to verify that the maps so defined are still simple random variables. Let $B_1$ and $B_2$ be the finite partitions of $\Omega$ associated with $\Delta_1$ and $\Delta_2$ respectively. Then $B = B_1 \cap B_2$ is the partition of $\Omega$ whose blocks are the pairwise intersections of blocks from $B_1$ and $B_2$. Clearly, the map $\Delta_1 \lor \Delta_2$ is constant on each block of $B$, hence a simple random variable. If further $\Delta$ is defined relative to a partition $B$ of $\Omega$, then $x(\Delta)$ is also constant on the blocks
of \( B \), hence also a simple random variable. Obviously, \((R_s, D)\) becomes an information algebra with these operations. The bottom element is the simple random variable \( E \) defined by \( E(\omega) = 0 \), the top element the simple random variable \( Z \) defined by \( Z(\omega) = 1 \) for all \( \omega \in \Omega \). Furthermore, for every \( \phi \in \Phi \) the map \( D_\phi(\omega) = \phi \), for all \( \omega \in \Omega \), is a simple random variable. By the mapping \( \phi \mapsto D_\phi \) the information algebra \((\Phi, D)\) is embedded into the information algebra \((R_s, D)\).

Note that the partial order in \( R_s \) is also defined point-wise such that \( \Delta_1 \leq \Delta_2 \) in \( R_s \) if, and only if, \( \Delta_1(\omega) \leq \Delta_2(\omega) \) for all \( \omega \in \Omega \).

There are two important special classes of simple random variables: If for a random variable \( \Delta \) defined relative to a partition \( B = \{B_1, \ldots, B_n\} \) it holds that \( \phi_i \neq \phi_j \) for \( i \neq j \), the variable is called canonical. It is a simple matter to transform any random variable \( \Delta \) into an associated canonical one: Take the union of all elements \( B_i \in B \) with identical values \( \phi_i \). This yields a new partition \( B' \) of \( \Omega \). Define \( \Delta'(\omega) = \Delta(\omega) \). Then \( \Delta' \) is the canonical version of \( \Delta \) and we write \( \Delta' = \Delta \rightarrow \). We may consider the set of canonical random variables, \( R_{s,c} \), and define between elements of this set combination and extraction as follows:

\[
\Delta_1 \lor_c \Delta_2 = (\Delta_1 \lor \Delta_2) \rightarrow, \\
x_c(\Delta) = (x(\Delta)) \rightarrow.
\]

Then \((R_{s,c}, D)\) is still an information algebra under these modified operations. We remark also that \((\Delta_1 \lor \Delta_2) \rightarrow = (\Delta_1 \rightarrow \lor \Delta_2 \rightarrow) \rightarrow \) and \((x(\Delta)) \rightarrow = (x(\Delta \rightarrow)) \rightarrow\). In fact, \((R_{s,c}, D)\) is the quotient algebra of \((R_s, D)\) relative to the congruence \( \Delta_1 \equiv \Delta_2 \), if \( \Delta_1 \rightarrow = \Delta_2 \rightarrow \).

Secondly, if \( \Delta(\omega) = 1 \) with probability zero, then \( \Delta \) is called normalized. We can associate a normalized random variables \( \Delta' \) with any simple random variable \( \Delta \) provided \( \Delta(\omega) \neq 1 \) occurs with a positive probability. In fact, let \( \Omega^\perp = \{\omega \in \Omega : \Delta(\omega) \neq 1\} \). This is a measurable set with probability \( P(\Omega^\perp) = 1 - P(\{\omega \in \Omega : \Delta(\omega) = 1\}) > 0 \). We consider then the new probability space \((\Omega, A, P')\), where \( P' \) is the conditional probability measure on \( A \) defined by

\[
P'(A) = \frac{P(A \cap \Omega^\perp)}{P(\Omega^\perp)}, \tag{9.1}
\]

if \( A \cap \Omega^\perp \neq \emptyset \) and \( P'(A) = 0 \), otherwise. On this new probability space define \( \Delta'(\omega) = \Delta(\omega) \). Clearly, it holds that \( (\Delta \rightarrow)^\perp = (\Delta') \rightarrow \).

The idea behind normalization becomes clear, when we consider combination of random variables: Each of two (normalized) random variables \( \Delta_1 \) and \( \Delta_2 \) represents some (uncertain) information with the following interpretation: One of the \( \omega \in \Omega \) must be the correct, but unknown assumption. However, if \( \omega \) happens to be the correct assumption, then under the first random variable information \( \Delta_1(\omega) \) can be asserted, and under the second
variable information $\Delta_2(\omega)$. Thus, together, still under the assumption $\omega$, information $\Delta_1(\omega) \vee \Delta_2(\omega)$ can be asserted. However, it is possible that $\Delta_1(\omega) \vee \Delta_2(\omega) = 1$, even if both $\Delta_1$ and $\Delta_2$ are normalized. But the element $1$ represents a contradiction. Thus in view of the information given by the variables $\Delta_1$ and $\Delta_2$ the assumption $\omega$ can not hold, it can (and must) be excluded. This amounts to normalize the random variable $\Delta_1 \vee \Delta_2$, by excluding all $\omega \in \Omega$ for which the combination results in a contradiction, and then to condition (i.e. normalize) the probability on non-contradictory assumptions. We refer to (Kohlas & Monney, 1995; Haenni et al., 2000) for a discussion and justification of these issues.

Two partitions $B_1$ and $B_2$ of $\Omega$ are called independent, if $B_{1,i} \cap B_{2,j} \neq \emptyset$ for all blocks $B_{1,i} \in B_1$ and $B_{2,j} \in B_2$. If furthermore $P(B_{1,i} \cap B_{2,j}) = P(B_{1,i}) \cdot P(B_{2,j})$ for all those pairs of blocks, then the two partitions $B_1$ and $B_2$ are called stochastically independent. In addition, if $\Delta_1$ and $\Delta_2$ are two simple random variables defined on these two partitions respectively, then these random variables are called stochastically independent too. Note that if $\Delta_1$ and $\Delta_2$ are stochastically independent, then their canonical versions $\Delta_1^\rightarrow$ and $\Delta_2^\rightarrow$ are stochastically independent too.

### 9.3 Support and Plausibility

This section is devoted to the study of the probability distribution of simple random variables. The starting point is the following question: Given a simple random variable $\Delta$ on a information algebra and an element $\phi \in \Phi$, under what assumptions can the information $\phi$ be asserted to hold? And how likely is it, that these assumptions are valid?

If $\omega \in \Omega$ is an assumption such that $\Delta(\omega) \geq \phi$, then $\Delta(\omega)$ implies $\phi$. In this case we may say that $\omega$ is an assumption supporting $\phi$, in view of the information conveyed by $\Delta$. Therefore we define for every $\phi \in \Phi$ the set

$$qs_\Delta(\phi) = \{ \omega \in \Omega : \phi \leq \Delta(\omega) \}$$

of assumptions supporting $\phi$. However, if $\Delta(\omega) = 1$, then $\omega$ is supporting every $\phi \in \Phi$, since $\phi \leq 1$. The null element $1$ represents the contradiction, which implies everything. In a consistent theory, contradictions must be excluded. Thus, we conclude that assumptions such that $\Delta(\omega) = 1$ are not really possible assumptions and must be excluded. Let

$$qs_\Delta(1) = \{ \omega \in \Omega : \Delta(\omega) = 1 \}.$$

We assume that $qs_\Delta(1)$ is not equal to $\Omega$; otherwise $\Delta$ is representing fully contradictory “information”. In other words, we assume that proper information is never fully contradictory. If we eliminate the contradictory assumptions from $qs(\phi)$, we obtain the support set

$$s_\Delta(\phi) = \{ \omega \in \Omega : \phi \leq \Delta(\omega) \neq 1 \} = qs_\Delta(\phi) - qs_\Delta(1).$$
CHAPTER 9. UNCERTAIN INFORMATION

of $\phi$, which is the set of assumptions properly supporting $\phi$ and the mapping $s_\Delta : \Phi \to 2^\Omega$ is called the allocation of support induced by $\Delta$. The set $qs(\phi)$ is called quasi-support set to underline that it contains contradictory assumptions. This set has little interest from a semantic point of view, but it is useful for technical and especially for computational purposes. These concepts capture the essence of probabilistic assumption-based reasoning in information algebras as discussed in more detail in (Kohlas & Monney, 1995; Haenni et al., 2000; Kohlas, 2003a) in a less general setting.

Here are the basic properties of allocations of support:

**Theorem 9.1** If $\Delta$ is a simple random variable on an information algebra $(\Phi, D)$, then the following holds for the associated allocation of support $qs_\Delta, s_\Delta$:

1. $qs_\Delta(0) = \Omega$, $s(1) = \emptyset$.

2. If $\Delta$ is normalized, then $qs_\Delta = s_\Delta$ and $qs_\Delta(1) = \emptyset$.

3. For any pair $\phi, \psi \in \Phi$,

$$qs_\Delta(\phi \lor \psi) = qs_\Delta(\phi) \cap qs_\Delta(\psi),$$

$$s_\Delta(\phi \lor \psi) = s_\Delta(\phi) \cap s_\Delta(\psi).$$

**Proof.** (1) and (2) follow immediately from the definition of the allocation of support. (3) follows since $\phi \lor \psi \leq \Delta(\omega)$ if, and only if $\phi \leq \Delta(\omega)$ and $\psi \leq \Delta(\omega)$. □

Knowing assumptions supporting a hypothesis $\phi$ is already interesting and important. It is the part logic can provide. On top of this, it is important to know how likely it is that a supporting assumption is valid. This is the part added by probability. If we know or may assume that the information is consistent, then we should condition the original probability measure $P$ in $\Omega$ on the event $qs_\Delta^c(1)$. This leads then to the probability space $(qs_\Delta^c(1), A \cap qs_\Delta^c(1), P')$, where $P'(A) = P(A)/P(qs_\Delta^c(1))$. The likelihood of supporting assumptions for $\phi \in \Phi$ can then be measured by

$$sp_\Delta(\phi) = P'(s_\Delta(\phi)).$$

The value $sp_\Delta(\phi)$ is called degree of support of $\phi$ associated with the random variable $\Delta$. The function $sp : \Phi \to [0, 1]$ is called the support function of $\Delta$. It corresponds to the distribution function of ordinary random variables.

It is for technical reasons convenient to define also the degree of quasi-support

$$qsp_\Delta(\phi) = P(s_\Delta(\phi)).$$
9.3. SUPPORT AND PLAUSIBILITY

Then, the degree of support can also be expressed in terms of degrees of quasi-support

\[ sp_\Delta(\phi) = \frac{qsp_\Delta(\phi) - qsp(1)}{1 - qsp_\Delta(1)}. \]

This is the form which is usually used in applications (Haenni et al., 2000).

In another consideration, we can also ask for assumptions \( \omega \in \Omega \), under which \( \Delta \) shows \( \phi \) to be possible, that is, not excluded, although not necessarily supported. If \( \Delta(\omega) \) is such that combined with \( \phi \) it leads to a contradiction, i.e. if \( \Delta(\omega) \lor \phi = 1 \), then under \( \omega \) the information \( \phi \) is excluded by a consistency consideration as above. So we define the set

\[ p_\Delta(\phi) = \{ \omega \in \Omega : \Delta(\phi) \lor \phi \neq 1 \}. \]

This is the set of assumptions under which \( \phi \) is not excluded, hence can be considered as possible. Therefore we call it the possibility set of \( \phi \). Note that \( p_\Delta(\phi) \subseteq qs_\Delta^c(1) \). We can then define the degree of possibility, also sometimes called degree of plausibility (e.g. in (Shafer, 1976)), by

\[ pl_\Delta(\phi) = P'(p_\Delta(\phi)). \]

If \( \omega \in qs_\Delta^c(1) - p_\Delta(\phi) \), then, under this assumption, \( \phi \) is impossible, is excluded. So the set \( qs_\Delta^c(1) - p_\Delta(\phi) \) contains arguments against \( \phi \) and

\[ do_\Delta(\phi) = P'(qs_\Delta^c(1) - p_\Delta(\phi)) = 1 - pl_\Delta(\phi). \]

can be called the degree of doubt into \( \phi \). Note that \( s_\Delta(\phi) \subseteq p_\Delta(\phi) \) since \( \phi \leq \Delta(\omega) \neq 1 \) implies \( \phi \lor \Delta(\omega) = \Delta(\omega) \neq 1 \). Hence, we see that for all \( \phi \in \Phi \) we have that \( sp_\Delta(\phi) \leq pl_\Delta(\phi) \). These consideration put simple random variables in the realm of the so-called Dempster-Shafer theory (Dempster, 1967b; Shafer, 1976).

To underline this, consider for a simple random variable \( \Delta \) with possible values \( \phi_1, \ldots, \phi_n \) the probabilities

\[ m(\phi_i) = \sum_{j : \phi_j = \phi_i} P(B_j). \]

Note that \( m(\phi_i) = P(B_i) \), if the random variable \( \Delta \) is canonical. Remark also that

\[ \sum_{i=1}^{n} m(\phi_i) = 1. \]

Such a collection of probabilities \( m(\phi_i) \) summing up to one for \( i = 1, \ldots, n \) is called a basic probability assignment (bpa). Since \( qs_\Delta(\phi) = \cup_{\phi \leq \phi_i} B_i \) and \( p_\Delta(\phi) = \cup_{\phi \lor \phi_i \neq 1} B_i \), we see that

\[ qs_\Delta(\phi) = \sum_{\phi \leq \phi_i} m(\phi_i), \quad pl_\Delta(\phi) = \sum_{\phi \lor \phi_i \neq 1} m(\phi_i). \]
So the bpa of a simple random variable determines its degrees of support and plausibilities. In (Shafer, 1976), support function are called belief functions. Furthermore, if $\Delta_1$ and $\Delta_2$ are two stochastically independent simple random variables with possible values $\phi_{1,1}, \ldots, \phi_{1,n}$ and $\phi_{2,1}, \ldots, \phi_{1,m}$, then the possible values of the combined random variable $\Delta = \Delta_1 \lor \Delta_2$ are $\phi_k$, where each $\phi_k$ is equal to a combination $\phi_{1,i} \lor \phi_{2,j}$. Therefore, the bpa of the combined variable $\Delta$ is

$$m(\phi_k) = \sum_{\phi_{1,i} \lor \phi_{2,j} = \phi_k} m_1(\phi_{1,i}) \cdot m_1(\phi_{2,j}).$$

If only normalized random variables are considered, then the combined variable $\Delta$ is to be normalized. Then, if

$$m(1) = \sum_{\phi_{1,i} \lor \phi_{2,j} = 1} m_1(\phi_{1,i}) \cdot m_1(\phi_{2,j}) < 1,$$

we obtain the normalized bpa of $\Delta^\downarrow$ as

$$m^\downarrow(\phi_k) = \frac{\sum_{\phi_{1,i} \lor \phi_{2,j} = \phi_k} m_1(\phi_{1,i}) \cdot m_1(\phi_{2,j})}{1 - m(1)} \quad (9.2)$$

So, bpa are also sufficient to compute the bpa of the combination of statistically independent pieces of uncertain information. This has been proposed in (Dempster, 1967a) and the formula (9.2) is therefore also called Dempster’s rule. (Shafer, 1976) took up Dempster’s theory and proposed “A Mathematical Theory of Evidence” where bpa and Dempster’s rule play an import role. In both theories the concept of a bpa is central. Although Dempster’s interpretation of the theory and Shafer’s are not quite the same, one speaks often of the Dempster-Shafer Theory. At least the underlying mathematics in both views are identical. We shall argue in this chapter that our present theory is a natural generalization of Dempster-Shafer theory which was confined essentially to finite subset algebras and simple random variables (in our terminology). However bpa can no more play the same basic role as in Dempster-Shafer theory, since bpa works only of simple random variables, but not for more general uncertain information. Also, the full flavor of the duality relation between support and plausibility is deployed only in the case of Boolean information algebras, see Section 10.6.

### 9.4 Random Mappings

When we want to go beyond simple random mappings, there are several ways to do this. The most radical is to consider any mapping $h : \Omega \rightarrow \Phi$ from a probability space $(\Omega, \mathcal{A}, P)$ into an information algebra $(\Phi, D)$. Let’s call such mappings random mappings in $(\Phi, D)$. As before, in the case of simple random variables, we may define the operations of combination and extraction between random mappings point-wise in $(\Phi, D)$:
1. **Combination:** Let $\Gamma_1$ and $\Gamma_2$ be two random mappings into $(\Phi, D)$, then $\Gamma_1 \lor \Gamma_2$ is the random mapping into $(\Phi, D)$ defined by
\[
(\Gamma_1 \lor \Gamma_2)(\omega) = \Gamma_1(\omega) \lor \Gamma_2(\omega).
\]
(9.3)

2. **Extraction:** Let $\Gamma$ be a random mapping into $(\Phi, D)$ and $x \in D$, then $x(\Gamma)$ is the random mapping into $(\Phi, D)$ defined by
\[
x(\Gamma)(\omega) = x(\Gamma(\omega)).
\]
(9.4)

For a fixed probability space $(\Omega, A, P)$, let $R_\Phi$ denote the set of all random mappings in $(\Phi, D)$. With the two operations defined above, $(R_\Phi, D)$ becomes an information algebra. The mapping $E(\omega) = 0$ for all $\omega \in \Omega$ is the neutral element of combination, the bottom element in $R_\Phi$; the map $Z(\omega) = 1$ the top element. It is obvious that $\Gamma' \leq \Gamma$ if and only if $\Gamma'(\omega) \leq \Gamma(\omega)$ for all $\omega \in \Omega$.

Consider the ideal completion $(I_{R_\Phi}, D)$ of the information algebra $(R_\Phi, D)$ of random mappings. Any element $\Gamma \in I_{R_\Phi}$ may be represented as
\[
\Gamma = \bigvee \{\Delta : \Delta \in R_\Phi, \Delta \leq \Gamma\},
\]
Obviously, we also have in the ideal completion $I_\Phi$ of $\Phi$
\[
\Gamma(\omega) = \bigvee \{\Delta(\omega) : \Delta \in R_\Phi, \Delta \leq \Gamma\} = \bigvee \{\phi : \phi \in \Phi, \phi \leq \Gamma(\omega)\}
\]
for all $\omega \in \Omega$. Therefore, $\Gamma$ is also a random mapping into the ideal completion $(I_\Phi, D)$ of the information algebra $(\Phi, D)$. And any random mapping into $I_\Phi$ belongs to $I_{R_\Phi}$. This shows that $(R_{I_\Phi}, D)$ is identical to the ideal completion $(I_{R_\Phi}, D)$ of the algebra $(R_\Phi, D)$.

As in the case of simple random variables (Section 9.3) we may define the allocation of support $s_\Gamma$ of a random mapping by
\[
s_\Gamma(\phi) = \{\omega \in \Omega : \phi \leq \Gamma(\omega)\}.
\]
(9.5)

We do not any more distinguish here between the semantic categories of support and quasi-support as in Section 9.3 and speak simply of support, even though (9.5) is strictly speaking a quasi-support.

This support, as defined in (9.5), has the same properties as the support of simple random variables, in particular, as in Theorem 9.1, $s_\Gamma(\emptyset) = \Omega$ and $s_\Gamma(\phi \lor \psi) = s_\Gamma(\phi) \cap s_\Gamma(\psi)$. Again, as before, with simple random variables, we may try to define the degree of support induced by a random mapping $\Gamma$ of a piece of information $\phi$ by
\[
s_{s_\Gamma}(\phi) = P(s_\Gamma(\phi)).
\]
(9.6)

This probability however is only defined if $s_\Gamma(\phi) \in A$. There is no guarantee that this holds in general. The only element which we know for sure to
be measurable is $s_{\Gamma}(\emptyset) = \Omega$. A simple way out of this problem would be to restrict random mappings to mappings $\Gamma$ for which $s_{\Gamma}(\phi) \in \mathcal{A}$ for all $\phi \in \Phi$ or even for all elements of the ideal completion $I_{\Phi}$. However, there is a priori no reason why we should restrict ourselves exactly to those mappings. Therefore we prefer other, more rational approaches to overcome the difficulty of an only partial definition of degrees of support. Here we propose a first solution. Later we present some alternatives.

(Shafer, 1979) advocates the use of probability algebras instead of probability spaces as a natural framework for studying belief functions. Since degrees of support are like belief functions (see Section 9.3) we adapt this idea here. First, we introduce the probability algebra associated with a probability space (Kappos, 1969). Let $\mathcal{J}$ be the $\sigma$-ideal of $\mathcal{P}$-null sets in the $\sigma$-algebra $\mathcal{A}$ of the probability space. Two sets $A', A'' \in \mathcal{A}$ are equivalent modulo $\mathcal{J}$, if $A' - A'' \in \mathcal{J}$ and $A'' - A' \in \mathcal{J}$. This means that the two sets have the same probability measure $P(A') = P(A'')$. This equivalence is a congruence in the Boolean algebra $\mathcal{A}$. Hence the quotient algebra $\mathcal{B} = \mathcal{A}/\mathcal{J}$ is a Boolean $\sigma$-algebra too. If $[A]$ denotes the equivalence class of $A$, then, for any countable family of sets $A_i, i \in I$,

$$[A]^c = [A^c],$$

$$\bigvee_{i \in I} [A_i] = \left[ \bigcup_{i \in I} A_i \right],$$

$$\bigwedge_{i \in I} [A_i] = \left[ \bigcap_{i \in I} A_i \right].$$

(9.7)

So $[A]$ defines a Boolean homomorphism from $\mathcal{A}$ onto $\mathcal{B}$, called projection. We denote $[\Omega]$ by $\top$ and $\emptyset$ by $\bot$. These are of course the top and bottom elements of $\mathcal{B}$. Now, as is well known, $\mathcal{B}$ has some further important properties (see (Halmos, 1963)): it satisfies the countable chain condition, which means that any family of disjoint elements of $\mathcal{B}$ is countable. Further, any Boolean algebra $\mathcal{B}$ satisfying the countable chain condition is complete. That is any subset $E \subseteq \mathcal{B}$ has a supremum $\vee E$ and an infimum $\wedge E$ in $\mathcal{B}$. Furthermore, the countable chain condition implies also that there is always a countable subset $D$ of $E$ with the same supremum and infimum, i.e. $\vee D = \vee E$ and $\wedge D = \wedge E$. We refer to (Halmos, 1963) for these results. Finally, by $\mu([A]) = P(A)$ a normalized, positive measure $\mu$ is defined on $\mathcal{B}$. Positive means here that $\mu(b) = 0$ implies $b = \bot$. A pair $(\mathcal{B}, \mu)$ of a Boolean $\sigma$-algebra $\mathcal{B}$, satisfying the countable chain condition, and a normalized, positive measure $\mu$ on it, is called a probability algebra.

We use now this construction of a probability algebra from a probability space to extend the definition of the degrees of support $s_{\Gamma}$ beyond elements $\phi$ for which $s_{\Gamma}(\phi)$ are measurable. Even if $s_{\Gamma}(\phi)$ is not measurable, any $A \in \mathcal{A}$ such that $A \subseteq s_{\Gamma}(\phi)$ represents an argument, that is a set of assumptions,
which supports \( \phi \). To exploit this remark, define for every set \( H \in \mathcal{P}(\Omega) \)
\[
\rho_0(H) = \bigvee \{ [A] : A \subseteq H, A \in \mathcal{A} \}. \tag*{(9.8)}
\]
This mapping has interesting properties as the following theorem shows.

**Theorem 9.2** The application \( \rho_0 : \mathcal{P}(\Omega) \to \mathcal{A}/\mathcal{J} \) as defined in (9.8) has the following properties:

\[
\begin{align*}
\rho_0(\Omega) &= \top, \\
\rho_0(\emptyset) &= \bot, \\
\rho_0 \left( \bigcap_{i \in I} H_i \right) &= \bigwedge_{i \in I} \rho_0(H_i). \tag*{(9.9)}
\end{align*}
\]
if \( \{H_i, i \in I\} \) is a countable family of subsets of \( \Omega \).

**Proof.** Clearly, \( \rho_0(\Omega) = [\Omega] = \top \in \mathcal{A}/\mathcal{J} \). Similarly, \( \rho_0(\emptyset) = [\emptyset] = \bot \in \mathcal{A}/\mathcal{J} \).

In order to prove the remaining identity, let \( H_i, i \in I \) be a countable family of subsets of \( \Omega \). For every index \( i \), there is a countable family of sets \( H'_j \in \mathcal{A} \) such that \( H'_j \subseteq H_i \) and \( \rho_0(H_i) = \bigvee \{ H'_j \} \) since \( \mathcal{B} \) satisfies the countable chain condition. Take \( A_i = \cup H'_j \). Then \( A_i \subseteq H_i \), \( A_i \in \mathcal{A} \) and \( P(A_i) = \mu(\rho_0(H_i)) \). Define \( A = \bigcap_{i \in I} A_i \in \mathcal{A} \). It follows that \( A \subseteq \bigcap_{i \in I} H_i \) and, because the projection is a \( \sigma \)-homomorphism, we obtain \( [A] = \bigwedge_{i \in I} [A_i] = \bigwedge_{i \in I} \rho_0(H_i) \).

We are going to show now that \( [A] = \rho_0(\bigcap_{i \in I} H_i) \) which proves then the theorem. For this it is sufficient to show that \( P(A) = \mu(\rho_0(\bigcap_{i \in I} H_i)) \). This is so, since \( P(A) = \mu([A]) \) and \( A \subseteq \bigcap_{i \in I} H_i \), hence \( [A] \leq \rho_0(\bigcap_{i \in I} H_i) \). Therefore, if \( \mu([A]) = \mu(\rho_0(\bigcap_{i \in I} H_i)) \) we must well have \( [A] = \rho_0(\bigcap_{i \in I} H_i) \), since \( \mu \) is positive.

Now, clearly \( P(A) \leq \mu(\rho_0(\bigcap_{i \in I} H_i)) \). As above, we conclude that there is an \( A' \in \mathcal{A}, A' \subseteq \bigcap_{i \in I} H_i \) such that \( P(A') = \mu(\rho_0(\bigcap_{i \in I} H_i)) \). Further, \( A' \cup (A - A') \subseteq \bigcap_{i \in I} H_i \) implies that \( P(A' \cup (A - A')) = P(A') \), hence \( P(A - A') = 0 \). Define \( A'_i = A_i \cup (A - A') \subseteq H_i \). Then \( A_i - A'_i = \emptyset \) and therefore,
\[
\begin{align*}
\mu(\rho_0(H_i)) &= P(A_i) \leq P(A'_i) = P(A_i) + P(A'_i - A_i) \\
&\leq \mu(\rho_0(H_i)). \tag*{(9.10)}
\end{align*}
\]
This implies that \( P(A'_i - A_i) = 0 \), therefore we have \( [A_i] = [A'_i] \). From this we obtain that
\[
\begin{align*}
\cap A'_i &= \cap (A_i \cup (A' - A)) = (A' - A) \cup (\cap A_i) \\
&= (A' - A) \cup A = A \cup A' = A' \cup (A - A')
\end{align*}
\]
But \( \cap A'_i \) and \( \cap A_i \) are equivalent, since \( \cap A'_i = \bigwedge [A'_i] = \bigwedge [A_i] = \cap [A_i] \). This implies finally that \( P(A) = P(\cap A_i) = P(\cap A'_i) = P(A') + P(A - A') = P(A') = \mu(\rho_0(\bigcap_{i \in I} H_i)) \). This is what was to be proved.

Take now \( \mathcal{B} = \mathcal{A}/\mathcal{J} \) and consider the probability algebra \( (\mathcal{B}, \mu) \). Then we compose the allocation of support \( s \) from \( \Phi \) into the power set \( \mathcal{P}(\Omega) \) with
the mapping $\rho_0$ from $\mathcal{P}(\Omega)$ into $\mathcal{B}$ to a mapping $\rho = \rho_0 \circ s : \Phi \to \mathcal{B}$. Now we see that

\[
\rho(0) = \rho_0(s(0)) = \rho_0(\Omega) = \top,
\]

\[
\rho(\phi \lor \psi) = \rho_0(s(\phi \lor \psi)) = \rho_0(s(\phi) \cap s(\psi)) = \rho_0(s(\phi)) \land \rho_0(s(\psi)) = \rho(\phi) \land \rho(\psi).
\]  

(9.11)

A mapping $\rho$ satisfying these two properties is called an allocation of probability (a.o.p.) on the information algebra $\Phi$. In fact, it allocates an element of the probability algebra $\mathcal{B}$ to any element of the algebra $\Phi$. In this way, a random mapping $\Gamma$ leads always to an allocation of probability $\rho_\Gamma = \rho_0 \circ s_\Gamma$, once a probability measure on the assumptions is introduced.

In particular, we may now define the degree of support for any $\phi \in \Phi$ by

\[
sp_\Gamma(\phi) = \mu(\rho_\Gamma(\phi)).
\]  

(9.12)

This extends the support function (9.6) to all elements $\phi$ of $\Phi$.

In this way, the degree of support $sp_\Gamma(\phi)$ is, according to (9.8), equal to the probability of the supremum of all $[A]$, where $A$ is measurable and supports $\phi$. This can be expressed also in another way. In order to see this, we note an important property of probability algebras: Clearly $\mu(\land b_i) \leq \inf_i \mu(b_i)$ and $\mu(\lor b_i) \geq \sup_i \mu(b_i)$ holds for any family of elements $\{b_i\}$. But there are important cases where equality hold (Halmos, 1963). A subset $D$ of $\mathcal{B}$ is called downward (upward) directed, if for every pair $b', b'' \in D$ there is an element $b \in D$ such that $b \leq b' \land b'' (b \geq b' \lor b'')$.

**Lemma 9.1** If $D$ is a downward (upward) directed subset of $\mathcal{B}$, then

\[
\mu(\land_{i \in D} b_i) = \inf_{i \in D} \mu(b_i), \quad (\mu(\lor_{i \in D} b_i) = \sup_{i \in D} \mu(b_i))
\]  

(9.13)

**Proof.** There is a countable subfamily of elements $b_i \in D$ which have the same meet. If $c_1, i = 1, 2, \ldots$ are the elements of this family, then define $c'_1 = c_1$ and select elements $c'_i$ in the downward directed set $D$ such that $c'_2 \leq c'_1 \land c_2, c'_3 \leq c'_2 \land c_3, \ldots$. Then $c'_1 \geq c'_2 \geq c'_3 \geq \ldots$ and this sequence has still the same infimum. However, by the continuity of probability we have

\[
\mu(\land b_i) = \mu(\land c'_i) = \lim_{i \to \infty} \mu(c'_i) \geq \inf_i \mu(b_i).
\]  

(9.14)

But this implies $\mu(\land b_i) = \inf_i \mu(b_i)$. The case of upwards directed sets is treated in the same way. \hfill \Box

Note now that $\{[A] : A \subseteq H, A \in \mathcal{A}\}$ is an upward directed family. Therefore, according to Lemma 9.1 we have

\[
sp_\Gamma(\phi) = \mu(\rho(\phi)) = \mu(\lor\{[A] : A \in \mathcal{A}, A \subseteq s_\Gamma(\phi)\})
\]

\[
= \sup\{\mu([A]) : A \in \mathcal{A}, A \subseteq s_\Gamma(\phi)\}
\]

\[
= \sup\{P(A) : A \in \mathcal{A}, A \subseteq s_\Gamma(\phi)\}
\]

\[
= P_\phi(s_\Gamma(\phi)),
\]  

(9.15)
where $P_\phi$ is the inner probability associated with $P$. This shows, that the degree of support of a piece of information $\phi$ defined by (9.12) is the inner probability of the support $s_\Gamma(\phi)$. Note that definitions (9.12) and (9.6) coincide, if $s_\Gamma(\phi) \in \mathcal{A}$. Support functions and inner probability measures are thus closely related. This result is very appealing: any measurable set $A$, which is contained in $s_\phi(\phi)$ supports $\phi$. So we expect $P(A) \leq sp_\Gamma(\phi)$. In the absence of further information, it is reasonable to take $sp_\Gamma(\phi)$ to be the least upper bound of the probabilities of $A$ supporting $\phi$.

A similar consideration can be made with respect to the possibility sets associated with elements of $\Phi$ with respect to a random mapping $\Gamma$. As in Section 9.3 we define the possibility set of $\phi$ as

$$p_\Gamma(\phi) = \{\omega \in \Omega : \Gamma(\omega) \lor \phi \neq 1\}.$$  

This set contains all assumptions $\omega$ which do not lead to a contradiction with $\phi$ under the mapping $\Gamma$. Thus, the probability of this set, if it is defined, measures the degree of possibility or the degree of plausibility of $\phi$,

$$pl_\Gamma(\phi) = P(p_\Gamma(\phi)).$$  

As in the case of the degree of support, there is no guarantee that $p_\Gamma(\phi)$ is measurable. But we can solve this problem in a way similar to the case of the degree of support. A measurable set $A \subseteq p_\Gamma^c$ can be seen as an argument against the hypothesis $\phi$, in particular, if $\Gamma$ is normalized. But $A \subseteq p_\Gamma^c$ is equivalent to $A^c \supseteq p_\Gamma$. So a measurable set $A \supseteq p_\Gamma$ can be considered as an argument, that hypothesis $\phi$ cannot be excluded. Therefore we define for every set $H \in \mathcal{P}$

$$\xi_0(H) = \wedge \{[A] : A \supseteq H, A \in \mathcal{A}\}. $$  

(9.17)

Note that $A \supseteq H$ if, and only if $A^c \subseteq H^c$. This implies that $\xi_0(H) = (\rho_0(H^c))^c$. From this in turn we conclude that the following corollary to Theorem 9.2 holds:

**Corollary 9.1** The application $\xi_0 : \mathcal{P}(\Omega) \rightarrow \mathcal{A}/\mathcal{J}$ as defined in (9.17) has the following properties:

$$\xi_0(\Omega) = \top,$$

$$\xi_0(\emptyset) = \bot,$$

$$\xi_0 \left( \bigcup_{i \in I} H_i \right) = \bigvee_{i \in I} \xi_0(H_i).$$  

(9.18)

if $\{H_i, i \in I\}$ is a countable family of subsets of $\Omega$.
As before we can now compose $p_T$ with $\xi_0$ to obtain a mapping $\xi_T = \xi_0 \circ p_T : \Phi \to B = A/J$. We may then define for any $\phi \in \Phi$ a degree of plausibility by

$$p{\Gamma}(\phi) = \mu(\xi_T(\phi)).$$

(9.19)

Using Lemma 9.1 we obtain also

$$p{\Gamma}(\phi) = \inf \{ P(A) : A \in A, A \supseteq p_T(\phi) \} = P^*(p_T(\phi)).$$

Here $P^*$ is the outer probability of the set $p_T(\phi)$. Thus, if $p_T(\phi)$ is measurable, then $P^*(p_T(\phi)) = P(p_T(\phi))$, which shows that (9.19) defines in fact an extension of the plausibility defined by (9.16).

In the general case considered here, no properties comparable to those of support (for instance Theorem 9.1) exist for possibility sets and degrees of possibility. This notion gets its full power only in the case of Boolean information algebra, where it becomes a dual concept to support (see Section 10.6).

### 9.5 Generalized Random Variables

Here and in the following section, we propose alternative approaches to define random variables in an information algebra. We start with the information algebra $(\mathcal{R}_s, D)$ of simple random variables with values in an information algebra $(\Phi, D)$ and defined on a sample space $(\Omega, A, P)$, and consider the ideal completion of this algebra, rather than the algebra $(\mathcal{R}_\Phi, D)$ considered in Section 9.4. Let $\mathcal{R}$ denote the ideal completion of $\mathcal{R}_s$. Then $(\mathcal{R}, D)$ is a $D$-compact information algebra with simple random variables $\mathcal{R}_s$ as finite elements (see Section 6, in particular Section 6.2). We call the elements of $\mathcal{R}$ generalized random variables. A generalized random variable is thus an ideal of simple random variables. As usual, we identify henceforth $\mathcal{R}_s$ with its image in $\mathcal{R}$ and the simple random variables $\Delta \in \mathcal{R}_s$ with their principal ideals $\downarrow \Delta$ in $\mathcal{R}$. Also we write $\Delta \leq \Gamma$ for $\Delta \in \Gamma$, referring to the order in $\mathcal{R}$. So, for any $\Gamma \in \mathcal{R}$ we may within the algebra $(\mathcal{R}, D)$ write $\Gamma = \vee\{ \Delta \in \mathcal{R}_s : \Delta \leq \Gamma \}$. Using the associativity of join in the complete lattice $\mathcal{R}$, we obtain also

$$\Gamma \vee \Gamma_2 = (\vee\{ \Delta_1 \in \mathcal{R}_s : \Delta_1 \leq \Gamma_1 \}) \vee (\vee\{ \Delta_2 \in \mathcal{R}_s : \Delta_2 \leq \Gamma_2 \})$$

$$= \vee\{ \Delta_1 \vee \Delta_2 : \Delta_1, \Delta_2 \in \mathcal{R}_s, \Delta_1 \leq \Gamma_1, \Delta_2 \leq \Gamma_2 \}. \quad (9.20)$$

In a similar way, by strong density (6.4), we find that

$$x(\Gamma) = x(\vee\{ \Delta \in \mathcal{R}_s : \Delta \leq \Gamma \}) = \vee\{ x(\Delta) : \Delta \in \mathcal{R}_s : \Delta \leq \Gamma \}. \quad (9.21)$$

To any $\Gamma \in \mathcal{R}$ we may associate a random mapping $\Gamma : \Omega \to J_\Phi$ from the underlying sample space into the ideal completion of $(\Phi, D)$ by defining

$$\Gamma(\omega) = \vee\{ \Delta(\omega) : \Delta \in \mathcal{R}_s, \Delta \leq \Gamma \}.$$
This random mapping is defined by a sort of pointwise limit within $I_\phi$. We denote the random mapping $\Gamma$ deliberately with the same symbol as the generalized random variable $\Gamma$. The reason is that the two concept can essentially be identified as the following lemmata show. Note that we denote join and information extraction for $x \in D$ with the same symbol in $R$ and in $I_\phi$.

**Lemma 9.2**

1. $\forall \Gamma_1, \Gamma_2 \in R$.

   $$(\Gamma_1 \lor \Gamma_2)(\omega) = \Gamma_1(\omega) \lor \Gamma_2(\omega) \text{ for all } \omega \in \Omega.$$ 

2. $\forall \Gamma \in R$ and $\forall x \in D$,

   $$(x(\Gamma))(\omega) = x(\Gamma(\omega)) \text{ for all } \omega \in \Omega.$$ 

**Proof.** (1) By definition we have

   $$(\Gamma_1 \lor \Gamma_2)(\omega) = \lor\{\Delta(\omega) : \Delta \leq \Gamma_1 \lor \Gamma_2\}$$

Consider now a $\phi \in (\Gamma_1 \lor \Gamma_2)(\omega)$. In the compact information algebra $(R, D)$ this means that $\phi \leq \lor\{\Delta(\omega) : \Delta \in \Gamma_1 \lor \Gamma_2\}$. The supremum on the right hand side is over a directed set in $I_\phi$. By compactness, there is therefore a $\Delta \leq \Gamma_1 \lor \Gamma_2$ such that $\phi \leq \Delta(\omega)$. Now, $\Delta \leq \Gamma_1 \lor \Gamma_2$ means by the definition of the join in $R$ that there is a $\Delta_1 \leq \Gamma_1, \Delta_1 \in R_s$ and a $\Delta_2 \leq \Gamma_2, \Delta_2 \in R_s$ such that $\Delta \leq \Delta_1 \lor \Delta_2$ (see Section 2.2). This implies that $\phi \leq (\Delta_1 \lor \Delta_2)(\omega) = \Delta_1(\omega) \lor \Delta_2(\omega)$, where $\Delta_1(\omega) \in \Gamma_1(\omega)$ and $\Delta_2(\omega) \in \Gamma_2(\omega)$. But this shows that $\phi \in \Gamma_1(\omega) \lor \Gamma_2(\omega)$.

Conversely, consider an element $\phi \in \Gamma_1(\omega) \lor \Gamma_2(\omega)$. By the definition of the join in $I_\phi$ this means that there are elements $\psi_1, \psi_2 \in \Phi$ such that $\phi \leq \psi_1 \lor \psi_2$, where $\psi_1 \leq \Gamma_1(\omega)$ and $\psi_2 \leq \Gamma_2(\omega)$. Now, $\psi_1 \leq \Gamma_1(\omega)$ means that $\psi_1 \leq \lor\{\Delta(\omega) : \Delta \leq \Gamma_1\}$. As above, by compactness, there is a $\Delta_1 \leq \Gamma_1$ such that $\psi_1 \leq \Delta_1(\omega)$. Similarly, there is a $\Delta_2 \leq \Gamma_2$ such that $\psi_2 \leq \Delta_2(\omega)$. Thus, $\phi \leq \Delta_1(\omega) \lor \Delta_2(\omega) = (\Delta_1 \lor \Delta_2)(\omega)$. Further $\Delta_1 \lor \Delta_2 \leq \Gamma_1 \lor \Gamma_2$. This implies $\phi \in (\Gamma_1 \lor \Gamma_2)(\omega)$, hence finally $(\Gamma_1 \lor \Gamma_2)(\omega) = \Gamma_1(\omega) \lor \Gamma_2(\omega)$.

(2) Assume next that $\phi \in (x(\Gamma))(\omega)$. As above, this implies that there is a $\Delta \in x(\Gamma)$ such that $\phi \leq \Delta(\omega)$. By the definition of $x(\Gamma)$ there is a $\Delta' \leq \Gamma$ such that $\Delta \leq x(\Delta')$. This implies $\phi \leq (x(\Delta'))(\omega) = x(\Delta'(\omega))$, which, together with $\Delta'(\omega) \leq \Gamma(\omega)$ shows that $\phi \in x(\Gamma(\omega))$.

Conversely, assume $\phi \leq x(\Gamma(\omega))$. Then $\phi \leq x(\psi)$ for some $\psi \in \Gamma(\omega)$. Again, as above, there is a $\Delta \leq \Gamma$ such that $\psi \leq \Delta(\omega)$. Therefore, we conclude that $\phi \leq x(\Delta(\omega)) = (x(\Delta))(\omega)$ and $x(\Delta) \leq x(\Gamma)$. This implies that $\phi \in (x(\Gamma))(\omega)$, hence $(x(\Gamma))(\omega) = x(\Gamma(\omega))$. □

According to this lemma we have a homomorphism between the algebra of generalized random variables and random mappings. In fact it is an embedding, since $\Gamma_1(\omega) = \Gamma_2(\omega)$ for all $\omega \in \Omega$ implies $\Gamma_1 = \Gamma_2$.

The next lemma, strengthens Lemma 9.2.
Lemma 9.3  If $X \subseteq \mathcal{R}$ is a directed set, then

$$(\lor_{\Gamma \in X} \Gamma)(\omega) = \lor_{\Gamma \in X} \Gamma(\omega).$$

Proof. If $\Gamma' \in X$, then $\Gamma' \leq \lor_{\Gamma \in X} \Gamma$, hence $\Gamma'(\omega) \leq (\lor_{\Gamma \in X} \Gamma)(\omega)$ and therefore

$$\lor_{\Gamma \in X} \Gamma(\omega) \leq (\lor_{\Gamma \in X} \Gamma)(\omega).$$

Conversely, assume $\phi \in (\lor_{\Gamma \in X} \Gamma)(\omega)$. Since

$$(\lor_{\Gamma \in X} \Gamma)(\omega) = \lor\{\Delta(\omega) : \Delta \leq \lor_{\Gamma \in X} \Gamma\}$$

we have $\phi \leq \Delta(\omega)$ for some simple random variable $\Delta \leq \lor_{\Gamma \in X} \Gamma$. Now, since $X$ is a directed set, by compactness, there is a $\Gamma' \in X$ such that $\Delta \leq \Gamma$, that is $\Delta(\omega) \leq \Gamma(\omega)$. It follows then that $\phi \leq \lor_{\Gamma \in X} \Gamma(\omega)$, which in turn implies

$$(\lor_{\Gamma \in X} \Gamma)(\omega) \leq \lor_{\Gamma \in X} \Gamma(\omega).$$

This concludes the proof of the lemma.

The theory of generalized random variables developed above may be presented in a similar way in the framework of compact information algebras. Here is a sketch of the approach:

Example 9.1 Generalized Random Variables in Compact Algebras. Let $(\Phi, D)$ be a compact information algebra with finite elements $\Phi_f$. We assume that $(\Phi_f, D)$ is a subalgebra of $(\Phi, D)$. Define then simple random variables $\Delta$ with finite elements from $\Phi_f$ as values. They form an information algebra $(\mathcal{R}_s; D)$ with combination and focusing defined pointwise. Order between simple random variables is also defined pointwise, i.e. $\Delta_1 \leq \Delta_2$ holds if, and only if $\Delta_1(\omega) \leq \Delta_2(\omega)$ for all sample points $\omega$ of the underlying probability space.

Since the ideal completion $(I_{\Phi_f}, D)$ of the information algebra $(\Phi_f, D)$ is isomorph to the compact algebra $(\Phi, D)$ (see Section 6.2), the theory above applies to the present case. Generalized random variables can thus be considered as random mappings with values in $\Phi$, defined as point-wise limits of simple random variables with finite elements as values.

As before, with random mappings in the previous Section 9.4, there is no guarantee that the support $s_{\Gamma}(\phi)$ of a generalized random variable $\Gamma$ is measurable for every $\phi \in \Phi$. But of course we can extend the support function to all of $\Phi$ by the allocation of probability as proposed in Section 9.4. However, we shall show later that the degrees of support $sp_{\Gamma}(\phi)$ of a generalized random variable $\Gamma$ is in fact determined by the degrees of support of its approximating simple random random variables $\Delta \leq \Gamma$ (see Section 9.8).
9.6 Random Variables

Information algebras are closed under finite combinations or joins. But there are information algebras which are closed under countable combination. In this section we consider such algebras and uncertain information relative to such algebras. Here follows the definition which will be used in the sequel:

**Definition 9.1 σ-Information Algebra** An information algebra \((\Phi,D)\) is called a σ-information algebra, if

1. **Countable Combination:** \(\Phi\) is closed under countable join.
2. **Continuity of Extraction:** For every monotone sequence \(\phi_1 \leq \phi_2 \leq \ldots \in \Phi\), and for any \(x \in D\), it holds that
   \[ x(\bigvee_{i=1}^{\infty} \phi_i) = \bigvee_{i=1}^{\infty} x(\phi_i). \]

The second condition is a natural extension of property (c) in item 3 of an existential quantor (see the axiom in Section 2.1). In fact, if \(\psi \leq \phi\), then this property (c), together with the idempotency of join gives \(x(\psi \vee \phi) = x(\phi) = x(\psi) \vee x(\phi)\). In this section, information algebras \((\Phi,D)\) are always assumed to be σ-algebras. There are many examples of σ-information algebras. First of all, any \(D\)-continuous or \(D\)-compact information \((\Phi,D)\) algebra is a σ-information algebra: Since in these cases \(\Phi\) is a complete lattice it is surely closed under countable join. The continuity of extraction follows from Theorems 6.3 and 6.19, since a monotone sequence is a directed set. A more specific example follows.

Important examples of σ-information algebras are the minimal extensions of an information algebra \((\Phi,D)\) which are closed under countable combination. This extension can be obtained using ideal completion. In order to do this we need to introduce a new concept. Let \((\Phi,D)\) be an information algebra and \((I_\Phi,D)\) its ideal completion. A subset \(S\) of \(I_\Phi\) is called σ-closed, if it is closed under countable combinations or joins. The intersection of any family of σ-closed sets is also σ-closed. Further the set \(I_\Phi\) itself is σ-closed. Therefore, for any subset \(X \subseteq I_\Phi\) we may define the σ-closure \(\sigma(X)\) as the intersection of all σ-closed sets containing \(X\).

We are particularly interested in \(\sigma(\Phi)\), the σ-closure of \(\Phi\) in \(I_\Phi\). Note that here, as in the sequel, we identify as usual \(\Phi\) with its embedding \(I(\Phi)\) under the mapping \(\phi \mapsto \downarrow \phi\) for simplicity of notation. Also we shall write \(\phi\), even if we operate within \(I_\Phi\). The σ-closure of \(\Phi\) can be characterized as follows:

**Theorem 9.3** If \((\Phi,D)\) is an information algebra, then

\[ \sigma(\Phi) = \{ I \in I_\Phi : I = \bigvee_{i=1}^{\infty} \psi_i, \psi_i \in \Phi, \forall i = 1, 2, \ldots \}. \]  

(9.22)
Proof. Clearly, the set on the right of equation (9.22) contains \( \Phi \) and is contained in \( \sigma(\Phi) \). We claim that this set is itself \( \sigma \)-closed. In fact, consider a countable set \( I_j \) of elements of this set, such that

\[
I_j = \bigvee_{i=1}^{\infty} \psi_{j,i}
\]

with \( \psi_{j,i} \in \Phi \). Using, the meanwhile well-known method, define the set \( J = \{(j,i) : j = 1, 2 \ldots ; i = 1, 2 \ldots \} \) and the sets \( J_j = \{(j,i) : i = 1, 2, \ldots \} \) for \( j = 1, 2, \ldots \), and \( K_i = \{(h,j) : 1 \leq h, j \leq i \} \) for \( i = 1, 2, \ldots \). Then we have

\[
J = \bigcup_{j=1}^{\infty} J_j = \bigcup_{i=1}^{\infty} K_i.
\]

By the laws of associativity in the complete lattice \( I_{\Phi} \) we obtain then

\[
\bigvee_{j=1}^{\infty} I_j = \bigvee_{j=1}^{\infty} (\bigvee_{(j,i) \in J_j} \psi_{j,i}) = \bigvee_{(j,i) \in J} \psi_{j,i} = \bigvee_{i=1}^{\infty} (\bigvee_{(h,j) \in K_i} \psi_{h,j}).
\]

But \( \bigvee_{(h,j) \in K_i} \psi_{h,j} \in \Phi \) for \( i = 1, 2, \ldots \). Hence \( \bigvee_{j=1}^{\infty} I_j \) belongs itself to the set on the right hand side of (9.22). This means that this set is indeed \( \sigma \)-closed. Since the set contains \( \Phi \), it contains also \( \sigma(\Phi) \), hence it equals \( \sigma(\Phi) \). \( \square \)

Consider now a monotone sequence \( \phi_1 \leq \phi_2 \leq \ldots \) of elements of \( \Phi \). Its supremum exists in \( I_{\Phi} \) and belongs in fact to \( \sigma(\Phi) \). The sequence is furthermore a directed set. Therefore, by Theorem 6.3 join commutes with information extraction, this is expressed in the following theorem. It shows continuity of extraction holds:

**Theorem 9.4** For a monotone sequence \( \phi_1 \leq \phi_2 \leq \ldots \) of elements of \( \Phi \), and for any \( x \in D \), we have in \( \sigma(\Phi) \) that

\[
x(\bigvee_{i=1}^{\infty} \phi_i) = \bigvee_{i=1}^{\infty} x(\phi_i). \tag{9.23}
\]

Theorem 9.4 shows in particular that \( \sigma(\Phi) \) is closed under focussing. In fact, if \( \phi_i \) is any sequence of elements of \( \Phi \), then we may define \( \psi_i = \bigvee_{k=1}^{\infty} \phi_k \in \Phi \), such that \( \psi_k \) for \( k = 1, 2, \ldots \) is a monotone sequence and \( I = \bigvee_{i=1}^{\infty} \phi_i = \bigvee_{i=1}^{\infty} \psi_i \). So, for \( I \in \sigma(\Phi) \) and any \( x \in D \) by Theorem 9.4

\[
x(I) = \bigvee_{i=1}^{\infty} x(\psi_i), \tag{9.24}
\]

where \( x(\psi_i) \in \Phi \) and hence \( x(I) \in \sigma(\Phi) \) by Theorem 9.3. As a \( \sigma \)-closed set \( \sigma(\Phi) \) is closed under combination. Therefore \( (\sigma(\Phi), D) \) is itself an information algebra, a subalgebra of \( (P_{\Phi}, D) \). Since it is closed under combination (i.e. join) of countable sets and satisfies condition (9.23) it is a \( \sigma \)-information algebra, the \( \sigma \)-algebra induced by \( (\Phi, D) \).
A particular and important case of such a construction is $(\sigma(\Phi_f), D)$ in a compact information algebra. Due to the Representation Theorem 6.7, this can be reduced to the situation of ideal completion, described above.

Simple random variables $\Delta$ on a probability space $(\Omega, \mathcal{A}, P)$ with values in an information algebra $(\Phi, D)$ are defined as in Section 9.2 and they form an information algebra $(\mathcal{R}_\sigma, D)$. We may define a random mapping $\Gamma : \Omega \to I_\Phi$ from a countable family of simple random variables $\Delta_i$ by

$$\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta_i(\omega).$$

We call such a random mapping $\Gamma$ a random variable in the algebra $(\Phi, D)$. Note that its values are in the ideal completion $I_\Phi$ of $\Phi$. In the case of a compact information algebra $(\Phi, D)$, the values of the simple random variables are considered to be finite, that is to be in $\Phi_f$.

Let now $\mathcal{R}_\sigma$ be the family of random variables in the $\sigma$-algebra $(\Phi, D)$.

**Lemma 9.4** A random variable $\Gamma$ is always the supremum of a monotone increasing sequence $\Delta_1 \leq \Delta_2 \leq \ldots$ of simple random variables, such that for all $\omega \in \Omega$,

$$\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta_i(\omega).$$

**Proof.** If $\Gamma$ is a random variable, then $\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta'_i(\omega)$ for some sequence $\Delta'_i$ of simple random variables. Define

$$\Delta_i = \bigvee_{j=1}^{i} \Delta'_j.$$

Then each $\Delta_i$ is a simple random variable, $i = 1, 2, \ldots$ and $\Delta_1 \leq \Delta_2 \leq \ldots$. From $\Delta'_i \leq \Delta_i$, we conclude that $\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta'_i(\omega) \leq \bigvee_{i=1}^{\infty} \Delta_i(\omega)$. On the other hand, $\Delta_i(\omega) \leq \Gamma(\omega)$, hence $\bigvee_{i=1}^{\infty} \Delta_i(\omega) \leq \Gamma(\omega)$, such that finally $\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta_i(\omega)$. □

Random variables are random mappings and as such can be combined and extracted point-wise in the ideal completion $(I_\Phi, D)$:

1. **Combination:** $(\Gamma_1 \vee \Gamma_2)(\omega) = \Gamma_1(\omega) \vee \Gamma_2(\omega)$,

2. **Focussing:** $x(\Gamma)(\omega) = x(\Gamma(\omega))$.

We have to verify that the resulting random mappings still belong to $\mathcal{R}_\sigma$, that is are random variables. So, let

$$\Gamma_1 = \bigvee_{i=1}^{\infty} \Delta_{1,i}, \quad \Gamma_2 = \bigvee_{i=1}^{\infty} \Delta_{2,i}.$$
Then we obtain, using associativity of the supremum
\[(\Gamma_1 \lor \Gamma_2)(\omega) = \Gamma_1(\omega) \lor \Gamma_2(\omega) = (\bigvee_{i=1}^{\infty} \Delta_{1,i}(\omega)) \lor (\bigvee_{i=1}^{\infty} \Delta_{2,i}(\omega)) = \bigvee_{i=1}^{\infty} (\Delta_{1,i}(\omega) \lor \Delta_{2,i}(\omega)) = \bigvee_{i=1}^{\infty} \Delta_{1,i}(\omega) \lor \bigvee_{i=1}^{\infty} \Delta_{2,i}(\omega) = \bigvee_{i=1}^{\infty} (\Delta_{1,i} \lor \Delta_{2,i})(\omega).\]

Since \(\Delta_{1,i} \lor \Delta_{2,i} \in \mathcal{R}\) this proves that \(\Gamma_1 \lor \Gamma_2 \in \mathcal{R}_\sigma\). Note then that, as usual, \(\Gamma_1 \leq \Gamma_2\) if and only if \(\Gamma_1(\omega) \leq \Gamma_2(\omega)\) for all \(\omega \in \Omega\), since random variables are random mappings.

Further, let
\[\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta_i(\omega),\]
where \(\Delta_i\) is an increasing sequence of simple random variables (see Lemma 9.4). Then, by the continuity of extraction,
\[x(\Gamma)(\omega) = x(\Gamma(\omega)) = x(\bigvee_{i=1}^{\infty} \Delta_i(\omega)) = \bigvee_{i=1}^{\infty} x(\Delta_i(\omega)) = \bigvee_{i=1}^{\infty} x(\Delta_i)(\omega) = \bigvee_{i=1}^{\infty} (\Delta_i \lor x(\Delta_i))(\omega).\]

Again, if \(\Delta_i\) are simple random variables, then so are the \(x(\Delta_i)\), therefore \(x(\Gamma)\) is indeed a random variable.

We expect \((\mathcal{R}_\sigma, D)\) to form an information algebra, even a \(\sigma\)-algebra. This is indeed true. We use the following lemma to prove this statement:

**Lemma 9.5** Assume \(\Gamma_i \in \mathcal{R}_\sigma\) for \(i = 1, 2, \ldots\) to be random variables. Then \(\bigvee_{i=1}^{\infty} \Gamma_i\) exists in the information algebra of random mappings into \(I_\Phi\), and for all \(\omega \in \Omega\),
\[\left(\bigvee_{i=1}^{\infty} \Gamma_i\right)(\omega) = \bigvee_{i=1}^{\infty} \Gamma_i(\omega)\]

**Proof.** Consider the random mapping \(\eta\) defined by \(\eta(\omega) = \bigvee_{i=1}^{\infty} \Gamma_i(\omega)\). Since \(\Gamma_i(\omega) \leq \bigvee_{i=1}^{\infty} \Gamma_i(\omega)\), it follows that \(\Gamma_i \leq \eta\), hence \(\eta\) is an upper bound of the random mappings \(\Gamma_i\). If \(\epsilon\) is another upper bounds, then \(\Gamma_i(\omega) \leq \epsilon(\omega)\), hence \(\eta(\omega) = \bigvee_{i=1}^{\infty} \Gamma_i(\omega) \leq \epsilon(\omega)\), therefore \(\eta \leq \epsilon\). Thus, \(\eta\) is the supremum of the random mappings \(\Gamma_i\). \(\square\)

**Theorem 9.5** The system \((\mathcal{R}_\sigma, D)\) of random variables, with combination and extraction defined as above forms a \(\sigma\)-information algebra.

**Proof.** As we have seen above, \(\mathcal{R}_\sigma\) is closed under combination (join) and extraction. The bottom element, the mapping \(E(\omega) = 0\) as well as the
9.6. RANDOM VARIABLES

Top element $Z(\omega) = 1$ belong also to $\mathcal{R}_\sigma$. So $(\mathcal{R}_\sigma, D)$ is a subalgebra of the algebra of random mappings, hence an information algebra.

We show that $(\mathcal{R}_\sigma, D)$ is $\sigma$-closed, that is, if $\Gamma_i \in \mathcal{R}_\sigma$ for $i = 1, 2, \ldots$, then $\bigvee_{i=1}^\infty \Gamma_i \in \mathcal{R}_\sigma$. Let

$$
\Gamma_j(\omega) = \bigvee_{i=1}^\infty \Delta_{j,i}(\omega), \text{ for } j = 1, 2, \ldots,
$$

where $\Delta_{j,i}$ are simple random variables, and define the random mapping $\Gamma$, using Lemma 9.5, by

$$
\Gamma(\omega) = \left( \bigvee_{i=1}^n \Gamma_i \right)(\omega) = \bigvee_{j=1}^\infty \Gamma_j(\omega) = \bigvee_{j=1}^\infty \left( \bigvee_{i=1}^\infty \Delta_{j,i} \right).
$$

Consider the sets $I_j = \{(j,i) : i = 1, 2, \ldots\}$ and

$$
I = \{(j,i) : j, i = 1, 2, \ldots\} = \bigcup_{j=1}^\infty I_j.
$$

Then we have

$$
\Gamma(\omega) = \bigvee_{(j,i) \in I} \Delta_{j,i}.
$$

Let further $K_i = \{(h,j) : h, j \leq i\}$ and define

$$
\Delta_i(\omega) = \bigvee_{h=1}^i \bigvee_{j=1}^h \Delta_{h,j}(\omega).
$$

Note that the $\Delta_i$ are simple random variables. Further we have

$$
I = \bigcup_{i=1}^\infty K_i
$$

and therefore, by the associativity of supremum,

$$
\Gamma(\omega) = \bigvee_{i=1}^\infty \left( \bigvee_{(h,j) \in K_i} \Delta_{h,j}(\omega) \right) = \bigvee_{i=1}^\infty \Delta_i(\omega).
$$

Thus the random mapping $\Gamma$ is indeed a random variable and $\mathcal{R}_\sigma$ is closed under countable combination.

It remains to verify the continuity of extraction. Assume $\Gamma_1 \leq \Gamma_2 \leq \ldots$ be a monotone sequence of random variables in $\mathcal{R}_\sigma$ and $x \in D$. Then,
the continuity of extraction in \((\mathcal{R}_\sigma, D)\) follows from this property in \((\Phi, D)\), using Lemma 9.5 and the continuity of extraction in \(\mathcal{R}_\sigma\), as follows:

\[
x(\bigvee_{i=1}^\infty \Gamma_i)(\omega) = x(\bigvee_{i=1}^n \Gamma_i)(\omega) = x(\bigvee_{i=1}^\infty \Gamma_i(\omega)) = \bigvee_{i=1}^\infty x(\Gamma_i(\omega)).
\]

So, we see that \(x(\bigvee_{i=1}^\infty \Gamma_i) = \bigvee_{i=1}^n x(\Gamma_i)\). This ends the proof. \(\square\)

Certainly, \((\mathcal{R}_s, D)\) is a subalgebra of \((\mathcal{R}_\sigma, D)\). Within the algebra \((\mathcal{R}_\sigma, D)\), each element of \(\mathcal{R}_\sigma\) is the supremum of the simple random variables it dominates as the following lemma shows.

**Lemma 9.6** Let \(\Gamma \in \mathcal{R}_\sigma\), defined by

\[
\Gamma(\omega) = \bigvee_{i=1}^\infty \Delta_i(\omega).
\]

Then, in the information algebra \((\mathcal{R}_\sigma, D)\)

\[
\Gamma = \bigvee_{i=1}^\infty \Delta_i = \bigvee \{\Delta : \Delta \in \mathcal{R}_s, \Delta \leq \Gamma\}.
\]  

(9.25)

**Proof.** The first equality in (9.25) follows directly from the definition of \(\Gamma\). Trivially, \(\Gamma\) is an upper bound of the set \(\{\Delta : \Delta \leq \Gamma\}\). If \(\Gamma'\) is another upper bound of this set, then it is also an upper bound of the \(\Delta_i\), hence \(\Gamma \leq \Gamma'\). Therefore, \(\Gamma\) is the least upper bound of the set \(\{\Delta : \Delta \leq \Gamma\}\). \(\square\)

This lemma shows that a random variable is also generalized random variable.

We now take the \(\sigma\)-closure of \(\mathcal{R}_s\) in the compact algebra \((\mathcal{R}, D)\) of generalized random variables. According to Theorem 9.3 elements of \(\sigma(\mathcal{R}_s)\) are defined as

\[
\Gamma = \bigvee_{i=1}^\infty \Delta_i, \text{ with } \Delta_i \in \mathcal{R}_s, \forall I = 1, 2, \ldots.
\]

Then \((\sigma(\mathcal{R}_s), D)\) is a \(\sigma\)-information algebra, containing \(\mathcal{R}_s\), i.e. the simple random variables. To random variables we can associate random mappings, just as with generalized random variables, defined by

\[
\Gamma(\omega) = \bigvee_{i=1}^\infty \Delta_i(\omega), \text{ with } \Delta_i \in \mathcal{R}_s, \forall I = 1, 2, \ldots.
\]

Note that \(\Gamma(\omega) \in \sigma(\Phi)\) by Theorem 9.3. Therefore, the elements of \(\sigma(\mathcal{R}_s)\) are random variables with values in the information algebra \((\sigma(\Phi), D)\). These random variables are thus within the ideal completion \(I_\Phi\) of \(\Phi\) pointwise limits of sequences of simple random variables. According to Lemma 9.2
combination and focusing can also be defined pointwise within the algebra \((I_\Phi, D)\), or more precisely, within the \(\sigma\)-information algebra \(\sigma(\Phi, D)\). As we have done above. This shows the equivalence of taking the \(\sigma\)-closure of \(\mathcal{R}_\ast\) and the definition of random variables as suprema of sequences of simple random variables.

### 9.7 Allocations of Probabilities

In Section 9.4 we have introduced the concept of an allocation of probability (a.o.p.) as a means to extend the degrees of support of a random mapping beyond the measurable elements \(\phi\), i.e. the elements for which \(s_\Gamma(\phi) \in \mathcal{A}\). These allocations of probability play an important role in the theory of random information. Therefore, we start here with a study of this concept, first independent of its relation to random mappings and random variables. In the subsequent Section 9.8 we examine the relation between random mappings and their associated allocations of probability.

Random mappings, and in particular generalized random variables and random variables, provide means to model explicitly the mechanisms which generate uncertain information. We refer to (Kohlas & Monney, 1995; Haenni et al., 2000; Kohlas, 2003a; Kohlas & Monney, 2007; Pouly & Kohlas, 2011) for more specific applications of this idea. Alternatively, allocations of probability may serve to directly assign beliefs to pieces of information. This is more in the spirit of a subjective, epistemological description of belief, advocated especially by G. Shafer (Shafer, 1973; Shafer, 1976; Shafer, 1979). In this view, allocations of probability are taken as the primitive elements, rather than random variables or hints. This is the point of view developed in this section (see also (Kohlas, 1997; Kohlas, 2003b)).

Consider an information algebra \((\Phi, D)\) and a probability algebra \((\mathcal{B}, \mu)\). An allocation of probability (a.o.p.) is a mapping \(\rho : \Phi \rightarrow \mathcal{B}\) such that

- \((A1)\) \(\rho(\mathbf{0}) = \top\),
- \((A2)\) \(\rho(\phi \lor \psi) = \rho(\phi) \land \rho(\psi)\).

If furthermore \(\rho(\chi) = \bot\) holds, then the allocation is called normalized. \((A1)\) says that the full belief is allocated to the trivial vacuous information. More important is \((A2)\). It says that the belief of a combined information \(\phi \lor \psi\) equals the common part of belief \(\rho(\phi) \land \rho(\psi)\) allocated to both of the two pieces of information \(\phi\) and \(\psi\). We remind that the a.o.p. derived from a random mapping satisfies these two properties (see (9.11)). Note, that if \(\phi \leq \psi\), that is, \(\phi \lor \psi = \psi\), then \(\rho(\phi \lor \psi) = \rho(\phi) \land \rho(\psi) = \rho(\psi)\), hence \(\rho(\psi) \leq \rho(\phi)\). A particular a.o.p. is defined by \(\nu(\phi) = \bot\), unless \(\phi = \mathbf{0}\), in which case \(\nu(\mathbf{0}) = \top\). This is called the vacuous allocation. It is associated with the vacuous information represented by the random mapping \(\Gamma(\omega) = \mathbf{0}\) for all \(\omega \in \Omega\).
We may think of an allocation of probability as the description of a body of belief relative to pieces information in an information algebra \((\Phi, D)\) obtained from a source of information. Two (or more) distinct sources of information will lead to the definition of two (or more) corresponding allocations of probability. Thus, in a general setting let \(A_\Phi\) be the set of all allocations of probability of \(\Phi\) in \((\mathcal{B}, \mu)\). Select two allocations \(\rho_i, i = 1, 2\) from \(A_\Phi\). How can they be combined in order to synthesize the two bodies of information they represent into a single, aggregated body?

The basic idea is as follows: Consider a piece of information \(\phi\) in \(\Phi\). If now \(\phi_1\) and \(\phi_2\) are two other pieces of information in \(\Phi\), such that \(\phi \leq \phi_1 \lor \phi_2\), then the common belief \(\rho_1(\phi_1) \land \rho_2(\phi_2)\) allocated to \(\phi_1\) and to \(\phi_2\) by the two allocations \(\rho_1\) and \(\rho_2\) respectively, is a belief allocated to \(\phi\) by the two allocations simultaneously. That is, the total belief \(\rho(\phi)\) to be allocated to \(\phi\) by the two allocations \(\rho_1\) and \(\rho_2\) together must equal at least the common belief allocated to \(\phi_1\) and \(\phi_2\) individually by each of the two allocations respectively, that is
\[
\rho(\phi) \geq \rho_1(\phi_1) \land \rho_2(\phi_2). \tag{9.26}
\]

In the absence of other information, it seems then reasonable to define the combined belief in \(\phi\), as obtained from the two sources of information, as the least upper bound of all these implied beliefs,
\[
\rho(\phi) = \lor\{\rho_1(\phi_1) \land \rho_2(\phi_2) : \phi \leq \phi_1 \lor \phi_2\}. \tag{9.27}
\]
This defines indeed a new allocation of probability:

**Theorem 9.6** The map \(\rho : \Phi \to \mathcal{B}\) as defined by (9.27) is an allocation of probability.

**Proof.** First, we have
\[
\rho(0) = \lor\{\rho_1(\phi_1) \land \rho_2(\phi_2) : 0 \leq \phi_1 \lor \phi_2\} = \rho_1(0) \land \rho_2(0) = \top.
\]
So (A1) is satisfied.

Next, let \(\psi_1, \psi_2 \in \Phi\). By definition we have
\[
\rho(\psi_1 \lor \psi_2) = \lor\{\rho_1(\phi_1) \land \rho_2(\phi_2) : \psi_1 \lor \psi_2 \leq \phi_1 \lor \phi_2\}.
\]
Now, \(\psi_1 \leq \psi_1 \lor \psi_2\) implies that
\[
\lor\{\rho_1(\phi_1) \land \rho_2(\phi_2) : \psi_1 \lor \psi_2 \leq \phi_1 \lor \phi_2\} \leq \lor\{\rho_1(\phi_1) \land \rho_2(\phi_2) : \psi_1 \leq \phi_1 \lor \phi_2\}
\]
and similarly for \(\psi_2\). Thus, we have \(\rho(\psi_1 \lor \psi_2) \leq \rho(\psi_1), \rho(\psi_2)\), that is \(\rho(\psi_1 \lor \psi_2) \leq \rho(\psi_1) \land \rho(\psi_2)\).
On the other hand,

\[
\{ (\phi_1, \phi_2) : \psi_1 \vee \psi_2 \leq \phi_1 \vee \phi_2 \}
\]

\[\supseteq \quad \{ (\phi_1, \phi_2) : \phi_1 = \phi_1' \vee \phi_1'', \phi_2 = \phi_2' \vee \phi_2'', \psi_1 \leq \phi_1' \vee \phi_2', \psi_2 \leq \phi_1'' \vee \phi_2'' \}.\]

By the distributive law for complete Boolean algebras we obtain then

\[
\rho(\psi_1 \vee \psi_2)
\]

\[\geq \quad \lor \{ \rho_1(\phi_1' \vee \phi_1'') \land \rho_2(\phi_2' \vee \phi_2'') : \psi_1 \leq \phi_1' \vee \phi_2', \psi_2 \leq \phi_1'' \vee \phi_2'' \}
\]

\[= \quad \lor \{ \rho_1(\phi_1') \land \rho_1(\phi_1'') \land \rho_2(\phi_2') \land \rho_2(\phi_2'') : \psi_1 \leq \phi_1' \vee \phi_2', \psi_2 \leq \phi_1'' \vee \phi_2'' \}
\]

\[= \quad (\lor \{ \rho_1(\phi_1') \land \rho_2(\phi_2') : \psi_1 \leq \phi_1' \vee \phi_2' \}) \land
\]

\[\quad (\lor \{ \rho_1(\phi_1'') \land \rho_2(\phi_2'') : \psi_2 \leq \phi_1'' \vee \phi_2'' \})
\]

\[= \quad \rho(\psi_1) \land \rho(\psi_2).
\]

(9.28)

This implies finally that \(\rho(\psi_1 \vee \psi_2) = \rho(\psi_1) \land \rho(\psi_2)\). Thus (A2) holds too and \(\rho\) is indeed an allocation of probability. \(\Box\)

In this way, in the set of allocations of probability \(A_N\), a binary combination operation is defined. We denote this operation for the time being by \(\otimes\). Thus, \(\rho\) as defined by (9.27) is written as \(\rho = \rho_1 \otimes \rho_2\). The following theorem gives us the elementary properties of this operation.

**Theorem 9.7** The combination operation, as defined by (9.27), is commutative, associative, idempotent and the vacuous allocation is the neutral element of this operation.

**Proof.** The commutativity of (9.27) is evident. For the associativity note that for a \(\phi \in \Phi\) we have, due to the associativity and distributivity of meet and join in complete Boolean algebras,

\[
((\rho_1 \otimes \rho_2) \otimes \rho_3)(\phi)
\]

\[= \quad \lor \{ (\rho_1 \otimes \rho_2)(\phi_1) \land \rho_3(\phi_3) : \phi \leq \phi_1 \lor \phi_3 \}
\]

\[= \quad \lor \{ \lor \{ \rho_1(\phi_1) \land \rho_2(\phi_2) : \phi_1 \leq \phi_1 \lor \phi_2 \} \land \rho_3(\phi_3) : \phi \leq \phi_1 \lor \phi_3 \}
\]

\[= \quad \lor \{ \rho_1(\phi_1) \land \rho_2(\phi_2) \land \rho_3(\phi_3) : \phi \leq \phi_1 \lor \phi_2 \lor \phi_3 \}
\]

For \((\rho_1 \otimes (\rho_2 \otimes \rho_3))(\phi)\) we obtain exactly the same result in the same way. This proves associativity.

To show idempotency consider

\[
(\rho \otimes \rho)(\phi) = \quad \lor \{ \rho(\phi_1) \land \rho(\phi_2) : \phi \leq \phi_1 \lor \phi_2 \}
\]

\[= \quad \lor \{ \rho(\phi_1 \lor \phi_2) : \phi \leq \phi_1 \lor \phi_2 \}
\]

since the last supremum is attained for \(\phi_1 = \phi_2 = \phi\).
Finally let \( \nu \) denote the vacuous allocation. Then, for any allocation \( \rho \) and any \( \Phi \) we have, noting that \( \nu(\psi) = \bot \), unless \( \psi = 0 \), in which case \( \nu(0) = \top \),

\[
(\rho \otimes \nu)(\phi) = \lor \{ \rho(\phi_1) \land \nu(\phi_2) : \phi \leq \phi_1 \lor \phi_2 \} = \rho(\phi).
\]

This shows that \( \nu \) is the neutral element for combination. \( \square \)

This theorem shows that \( A_\Phi \) is a semilattice. Indeed, a partial order between allocations can be introduced as usual by defining \( \rho_1 \leq \rho_2 \) if \( \rho_1 \otimes \rho_2 = \rho_2 \). This means that for all \( \phi \in \Phi \),

\[
\rho_1 \otimes \rho_2(\phi) = \lor \{ \rho_1(\phi_1) \land \rho_2(\phi_2) : \phi \leq \phi_1 \lor \phi_2 \} = \rho_2(\phi).
\]

We have therefore always \( \rho_1(\phi_1) \land \rho_2(\phi_2) \leq \rho_2(\phi) \) if \( \phi \leq \phi_1 \otimes \phi_2 \). Take now \( \phi_1 = \phi \) and \( \phi_2 = 0 \), such that \( \phi \leq \phi \otimes 0 = \phi \), to obtain \( \rho_1(\phi) \land \rho_2(\phi) = \rho_1(\phi) \leq \rho_2(\phi) \). Thus we have \( \rho_1 \leq \rho_2 \) if, and only if, \( \rho_1(\phi) \leq \rho_2(\phi) \) for all \( \phi \in \Phi \). Clearly, the combination \( \rho_1 \otimes \rho_2 \) is the supremum of the two a.o.p. in this order. Therefore we shall henceforth write \( \rho_1 \lor \rho_2 \) for this combination.

The vacuous a.o.p. is the bottom element of this semilattice. And the a.o.p. defined by \( \zeta(\phi) = \top \) for all information elements, is the top element to the semilattice \( A_\Phi \), so that \( \rho \lor \zeta = \zeta \). So the semilattice of a.o.p.s \( A_\Phi \) is a \( 0, 1 \) semilattice.

Next we turn to the operation of extracting a part of an allocation of probability according to an operator \( x \in D \). Let \( \rho \) be an allocation of probability on an information algebra \( (\Phi, D) \) and \( x \in D \). Just as it is possible to extract a part of a piece of information \( \phi \) from \( \Phi \) with the aid of the operator \( x \), it should also be possible to focus the belief represented by the a.o.p. \( \rho \) to the information supported by the domain \( x \). This means to extract the information related to \( x \) from \( \rho \). Thus, for a \( \phi \in \Phi \) consider the beliefs allocated to pieces of information \( \psi \) which are supported by \( x \) and which entail \( \phi \), i.e. \( \phi \leq \psi = x(\psi) \). The part of the belief allocated to \( \phi \) and related to the domain \( x \), \( x(\rho)(\phi) \) must then be at least \( \rho(\psi) \),

\[
x(\rho)(\phi) \geq \rho(\psi) \text{ for any } \psi = x(\psi) \geq \phi.
\]

In the absence of other information, it seems again, as above, reasonable to define \( x(\rho)(\phi) \) to be the least upper bound of all these implied supports,

\[
x(\rho)(\phi) = \lor \{ \rho(\psi) : \psi = x(\psi) \geq \phi \}.
\]

This defines indeed an allocation of probability:

**Theorem 9.8** The map \( x : \Phi \to B \) as defined by (9.30) is an allocation of probability.
Thus (A1) is verified.

9.7. ALLOCATIONS OF PROBABILITIES

163

associative laws of complete Boolean algebras, for any \( \phi \)
This proves property (A2) for an allocation of support.

We have to verify items (a) to (c) in point 3 of the Definition
of an information algebra. The top element in \( A \)

Theorem 9.9
For every \( x \in D \) the map \( x : A_\Phi \to A_\Phi \) is an existential quantifier.

Proof. We have by definition

Thus (A1) is verified.

Again by definition,

\[ x(\rho)(\phi_1 \lor \phi_2) = \lor \{\rho(\psi) : \psi = x(\psi) \geq \phi_1 \lor \phi_2\}. \]

From \( \phi_1, \phi_2 \leq \phi_1 \lor \phi_2 \) it follows that \( x(\rho)(\phi_1 \lor \phi_2) \leq x(\rho)(\phi_1), x(\rho)(\phi_2) \) and thus \( x(\rho)(\phi_1 \lor \phi_2) \leq x(\rho)(\phi_1) \land x(\rho)(\phi_2) \).

On the other hand,

\[ \{ \psi : \psi = x(\psi) \geq \phi_1 \lor \phi_2 \} \]
\[ \supseteq \{ \psi = \psi_1 \lor \psi_2 : \psi_1 = x(\psi_1) \geq \phi_1, \psi_2 = x(\psi_2) \geq \phi_2 \}. \]

From this we obtain, using the distributive law for complete Boolean algebras,

\[ x(\rho)(\phi_1 \lor \phi_2) \]
\[ \geq \lor \{\rho(\psi_1 \lor \psi_2) : \psi_1 = x(\psi_1) \geq \phi_1, \psi_2 = x(\psi_2) \geq \phi_2\} \]
\[ = \lor \{\rho(\psi_1) \land \rho(\psi_2) : \psi_1 = x(\psi_1) \geq \phi_1, \psi_2 = x(\psi_2) \geq \phi_2\} \]
\[ = (\lor \{\rho(\psi_1) : \psi_1 = x(\psi_1) \geq \phi_1\}) \land (\lor \{\rho(\psi_2) : \psi_2 = x(\psi_2) \geq \phi_2\}) \]
\[ = \rho(\psi_1) \land \rho(\psi_2). \]

This proves property (A2) for an allocation of support. \( \square \)

Next we show that extraction as defined in (9.30) determines existential quantifiers.

Theorem 9.9 For every \( x \in D \) the map \( x : A_\Phi \to A_\Phi \) is an existential quantifier.

Proof. We have to verify items (a) to (c) in point 3 of the Definition
of an information algebra. The top element in \( A_\Phi \) is the a.o.p. \( \zeta \). Now
by (9.30) we have \( x(\zeta)(\phi) = \top \) for all \( \phi \in \Phi \), hence \( x(\zeta) = \zeta \). This is (a).
Further, since \( \psi = x(\psi) \geq \phi \) implies \( \rho(\psi) \leq \rho(\phi) \), we conclude that
\( x(\rho)(\phi) \leq \rho(\phi) \), hence \( x(\rho) \leq \rho \). This is (b).

Item (c) is a bit more involved. By definition and the distributive and associative laws of complete Boolean algebras, for any \( \phi \in \Phi \)

\[ (x(\rho_1) \lor x(\rho_2))(\phi) = \lor \{x(\rho_1)(\phi_1) \land x(\rho_2)(\phi_2) : \phi \leq \phi_1 \lor \phi_2\} \]
\[ = \lor \{\lor \{\rho_1(\psi_1) : \psi_1 = x(\psi_1) \geq \phi_1\} \}
\[ \land (\lor \{\rho_2(\psi_2) : \psi_2 = x(\psi_2) \geq \phi_2\}) : \phi \leq \phi_1 \lor \phi_2\} \]
\[ = \lor \{\rho_1(\psi_1) \land \rho_2(\psi_2) : \psi_1 = x(\psi_1) \geq \phi_1, \psi_2 = x(\psi_2) \geq \phi_2, \phi \leq \phi_1 \lor \phi_2\} \]
\[ = \lor \{\rho_1(\psi_1) \land \rho_2(\psi_2) : \psi_1 = x(\psi_1), \psi_2 = x(\psi_2), \phi \leq \psi_1 \lor \psi_2\}. \]
Also by definition,
\[(x(\rho_1) \lor \rho_2)(\phi) = \lor\{x(\rho_1)(\phi_1) \land \rho_2(\phi_2) : \phi \leq \phi_1 \lor \phi_2\}\]

and therefore, again by definition, and the associative and distributive laws of complete Boolean algebras

\[x(\rho_1) \lor \rho_2(\phi) = \lor\{\lor\{x(\rho_1)(\phi_1) \land \rho_2(\phi_2) : \phi \leq \psi = x(\psi)\} : \phi \leq \psi = x(\psi) \leq \phi_1 \lor \phi_2\}\]

Now, whenever \(\psi_1 = x(\psi_1), \psi_2 = x(\psi_2), \phi \leq \psi_1 \land \psi_2\), define \(\psi = \psi_1 \lor \psi_2\). Then \(x(\psi) = \psi\) and \(\phi \leq \psi = x(\psi) \leq \psi_1 \lor \psi_2, \psi_1 = x(\psi_1)\). This shows that

\[x(\rho_1) \lor x(\rho_2)(\phi) \leq x(x(\rho_1) \lor \rho_2)(\phi).

On the other hand, whenever \(\psi_1 = x(\psi_1), \phi \leq \psi = x(\psi) \leq \psi_1 \lor \psi_2\), then \(\phi \leq \psi = x(\psi) \leq x(\psi_1 \lor \psi_2) = \psi_1 \lor x(\psi_2)\) and \(\psi_2 \geq x(\psi_2)\). Further, \(\psi_2 \geq x(\psi_2)\) implies that \(\rho_2(\psi_2) \leq \rho_2(x(\psi_2))\). Therefore (renaming \(x(\psi_2)\) as \(\psi_2\)), we obtain

\[x(x(\rho_1) \lor \rho_2)(\phi) \leq \lor\{\lor\{\rho_1(\psi_1) \land \rho_2(\psi_2) : \psi = x(\psi_1), \psi_2 = x(\psi_2), \phi \leq \psi = x(\psi) \leq \psi_1 \lor \psi_2\} : \psi_1 = x(\psi_1), \psi_2 = x(\psi_2), \phi \leq \psi_1 \lor \psi_2\},

since we may always take \(\psi = \psi_1 \lor \psi_2\) such that \(x(\psi) = x(\psi_1 \lor \psi_2) = \psi_1 \lor \psi_2\). This shows that

\[x(\rho_1) \lor x(\rho_2)(\phi) \geq x(\rho_1) \lor \rho_2)(\phi).

And this proves (c) of item 3 in Definition 2.1. So the map \(x : A_\Phi \to A_\Phi\) is an existential quantifier. \(\square\)

Finally, we show that the maps \(x : A_\Phi \to A_\Phi\) for \(x \in D\) form a commutative, idempotent semigroup, that is a semilattice.

**Theorem 9.10** The maps \(x : A_\Phi \to A_\Phi\) for \(x \in D\) form a commutative, idempotent semigroup under composition. For \(\rho \in A_\Phi\) and \(x, y \in D\),

\[y(x(\rho)) = (x \land y)(\rho).\]  

**Proof.** As maps for \(x \in D\), the \(x\) form a semigroup under composition. Commutativity and idempotency of the semigroup follow from (9.31). So
we prove this identity. Note that the meet in (9.31) is the meet defined in
the semilattice $D$ in the underlying information algebra $(\Phi, D)$. But, once
(9.31) is proved, it becomes also the meet between the maps $x : A_\Phi \to A_\Phi$, as
induced by the commutative, idempotent semigroup of maps $x : A_\Phi \to A_\Phi$.

By definition and the associative law for complete Boolean algebras

$$y(x(\rho))(\phi) = \vee\{\{\rho(\psi) : \rho = y(\psi) \geq \phi\}
= \{\rho(\psi) : \psi = y(x(\psi)) \geq \phi\}.\]$$

Also by definition, we have

$$(x \land y)(\rho) = \vee\{\rho(\psi) : \psi = y(x(\psi)) \geq \phi\}.\]$$

If $\psi$ is supported by $x \land y$, then it is supported both by $x$ and by $y$. Set here
now $\psi' = \psi$. Then it follows that $\psi' = x(\psi') \geq \psi = y(\psi) \geq \phi$. This implies that

$$y(x(\rho))(\phi) \geq (x \land y)(\rho)(\phi).$$

On the other hand, $\psi' = x(\psi') \geq \psi = y(\psi) \geq \phi$ implies that $y(\psi') = y(x(\psi')) \geq y(\psi) \geq \phi$. Furthermore, $y(\psi') \leq \psi'$ implies $\rho(\psi') \leq \rho(y(\psi'))$. Thus we obtain

$$y(x(\rho))(\phi) \leq \vee\{\rho(\psi) : \rho = y(\psi) = y(x(\psi')) \geq \phi\}.\]$$

Now put in this formula $y(\psi') = \psi$, such that $y(\psi) = \psi$. Then we obtain

$$y(x(\rho))(\phi) \leq \vee\{\rho(\psi) : \rho = y(\psi) = y(x(\psi)) \geq \phi\}
= \vee\{\rho(\psi) : \rho = y(\psi) = (x \land y)(\psi) \geq \phi\}
= (x \land y)(\rho)(\phi).\]$$

This proves (9.31). □

According to this result we write henceforth $x(\rho)$ instead of $x(\rho)$ and
identify in this way the two semilattices of operators $x$ for $\Phi$ and $A_\Phi$.
This will be further justified below. Theorems 9.7, 9.9 and 9.10 show that
$(A_\Phi, D)$, that is the family of allocations of probability on $\Phi$, form an information algebra.

We show now that the algebra $(A_\Phi, D)$ is in fact an extension of the
information algebra $(\Phi, D)$. Consider for any $\phi \in \Phi$ the the following map
of $\Phi$ into $B$:

$$\rho_\phi(\psi) = \begin{cases} \top & \text{if } \psi \leq \phi, \\ \bot & \text{otherwise,} \end{cases} \quad (9.32)$$
It allocates total belief to all elements of information implied by $\phi$, that is to all elements of the principal ideal $\downarrow \phi$, and no belief to all other elements. This map is clearly an allocation of probability, it is called a deterministic allocation. It is a degenerate allocation insofar as there is no uncertainty in the information it expresses. It states simply that the piece of information $\phi$ holds. Obviously the bottom a.o.p. $\nu = \rho_0$ is a deterministic allocations, and we add the top element $\zeta$ to the deterministic a.o.p.s. Now, for $\phi_1, \phi_2 \in \Phi$ we have

$$\rho_{\phi_1 \lor \phi_2}(\psi) = \lor\{\rho_{\phi_1}(\psi_1) \land \rho_{\phi_2}(\psi_2) : \psi \leq \psi_1 \lor \psi_2\} = \begin{cases} \top & \text{if } \psi \leq \phi_1 \lor \phi_2, \\ \bot & \text{otherwise} \end{cases} = \rho_{\phi_1 \lor \phi_2}(\psi). \quad (9.33)$$

So, the combination of deterministic allocations of $\phi_1$ and $\phi_2$ produces the deterministic a.o.p of $\phi_1 \lor \phi_2$. Note that also $\rho_\phi \lor \zeta = \zeta$.

Further, for any $\phi \in \Phi$,

$$x(\rho_\phi)(\psi) = \lor\{\rho_\phi(\psi') : \psi' = x(\psi') \geq \psi\}.$$ 

This equals $\top$, if there is a $\psi' = x(\psi') \geq \psi$ such that $\psi' \leq \phi$, and $\bot$ otherwise. But, we have $\psi' = x(\psi') \leq \phi$ if, and only if, $\psi' = x(\psi') \leq x(\phi)$. This shows that $x(\rho_\phi)(\psi) = \rho_{x(\phi)}(\psi)$, hence $x(\rho_\phi) = \rho_{x(\phi)}$. The extraction of a deterministic a.o.p. related to $\phi$ by $x$ yields the deterministic a.o.p. associated with $x(\phi)$.

The mapping $\phi \mapsto \rho_\phi$ is thus an embedding of $(\Phi, D)$ in $(A_\Phi, D)$. In this sense, $(A_\Phi, D)$ extends the information algebra $(\Phi, D)$.

These results are collected in the following theorem:

**Theorem 9.11** Let $(\Phi, D)$ be an information algebra. The the set of allocations of probability $A_\Phi$ of $\Phi$ into some probability algebra $(B, \mu)$ form also an information algebra $(A_\Phi, D)$ where combination is defined by (9.27) and extraction by (9.30). The information algebra $(\Phi, D)$ is embedded in $(A_\Phi, D)$.

In the next section we pursue by examining the question how random mappings and allocations of probability, and especially their respective information algebras, are related.

### 9.8 Random Mappings and Allocations

In Section 9.4 it has been shown that a random mapping generates an allocation of probability, which specifies how much belief, according to the information represented by the random mapping, is to be assigned to an element of $\Phi$. In this section the relations between random mappings and
allocations of probability will be examined in more detail. In particular, we address the question, whether the operations between random mappings, combination and extraction, are reflected in the corresponding operations of the associated a.o.p., in other words, whether the mapping $\Gamma \mapsto \rho_\Gamma$ is a homomorphism between random mappings and associated allocations of probability.

We start with simple random variables. Fix an information algebra $(\Phi, D)$ and a probability space $(\Omega, \mathcal{A}, P)$. For any simple random variable $\Delta \in \mathcal{R}_s$ defined on this probability space, we have seen that all elements of $\Phi$ and even of $I_\Phi$ have measurable allocations of support $s_\Delta(\phi) \in \mathcal{A}$ and their degree of support is well defined. If we pass in this case from the probability space $(\Omega, \mathcal{A}, P)$ to its associated probability algebra $(\mathcal{B}, \mu)$ (see Section 9.4), then we can define the allocation of probability (a.o.p.) associated with the random variable $\Delta$,

$$\rho_\Delta(\phi) = [s_\Delta(\phi)]$$

for all elements $\phi \in \Phi$ and even for all elements in $I_\Phi$. Thus, we obtain for the degree of support induced by the random variable $\Delta$,

$$sp_\Delta(\phi) = P(s_\Delta(\phi)) = \mu(\rho_\Delta(\phi)).$$

Again this holds on all of $\Phi$ and even its ideal completion $I_\Phi$. The mapping $\rho_\Delta : \Phi \to \mathcal{B}$ satisfies the properties (A1) and A(2) of an allocation of probability in the previous Section 9.7.

A simple random variable $\Delta$ is defined by a partition $\{B_1, \ldots, B_n\}$ of $\Omega$ consisting of measurable blocks $B_i$ and a mapping defined by $\Delta(\omega) = \phi_i$ for all $\omega \in B_i$, and $i = 1, \ldots, n$. We write $\Delta(\omega) = \Delta(B_i)$, if $\omega \in B_i$. To the partition $\{B_1, \ldots, B_n\}$ of $\Omega$ corresponds a partition $\{[B_1], \ldots, [B_n]\}$ of the probability algebra $\mathcal{B}$. That is, we have $[B_i] \wedge [B_j] = \perp$ if $i \neq j$, and $\vee_{i=1}^n [B_i] = \top$. The simple random variable $\Delta$ can also be defined by a mapping $\Delta([B_i]) = \phi_i$ from the partition of $\mathcal{B}$ into $\Phi$. Its allocation of probability can then also be determined as

$$\rho_\Delta(\phi) = \vee\{[B_i] : \phi \leq \Delta([B_i])\}. \quad (9.34)$$

We note that $\rho_\Delta = \rho_\Delta^\rightarrow$. So, as far as allocation of probability (and support) is concerned we might as well restrict ourselves to considering canonical simple random variables and their information algebra $(\mathcal{R}_{s,c}, D)$ (see Section 9.2).

We now consider the mapping $\rho : \Delta \mapsto \rho_\Delta$ which maps simple random variables into a.o.p.s We show that this mapping is a homomorphism:

**Theorem 9.12** Let $\Delta_1, \Delta_2, \Delta \in \mathcal{R}_s$ be simple random variables, defined on partitions in a probability algebra $(\mathcal{B}, \mu)$ with values in an information
algebra \((\Phi, D)\). Then, for all \(\phi \in \Phi\) and \(x \in D\),
\[
\rho_{\Delta_1 \lor \Delta_2}(\phi) = (\rho_{\Delta_1} \lor \rho_{\Delta_2})(\phi) \tag{9.35}
\]
\[
\rho_{x(\Delta)}(\phi) = x(\rho_{\Delta})(\phi). \tag{9.36}
\]

It is understood that in this theorem the join on the left is the one in the semilattice of simple random variables, whereas on the right it is the one in the semilattice of a.o.p.s. Similarly, the extraction operator \(x\) on the left is the one in the information algebra \((\mathcal{R}, D)\) of simple random variables, the one on the right is the one in the information algebra \((A_{\Phi}, D)\) of a.o.p.

**Proof.** (1) Assume that \(\Delta_1\) is defined on the partition \(\{B_{1,1}, \ldots, B_{1,n}\}\) and \(\Delta_2\) on the partition \(\{B_{2,1}, \ldots, B_{2,m}\}\) of \(\mathcal{B}\). From the definition of an allocation of probability, and the distributive and associative laws for complete Boolean algebra, we obtain

\[
(\rho_{\Delta_1} \lor \rho_{\Delta_2})(\phi)
= \lor\{\rho_{\Delta_1}(\phi_1) \land \rho_{\Delta_2}(\phi_2) : \phi \leq \phi_1 \lor \phi_2\}
= \lor\{(\lor\{B_{1,i} : \phi_1 \leq \Delta_1(B_{1,i})\}) \land (\lor\{B_{2,j} : \phi_2 \leq \Delta_2(B_{2,j})\}) : \phi \leq \phi_1 \lor \phi_2\}
= \lor\{\lor\{B_{1,i} \land B_{2,j} \neq \bot \land \phi_1 \leq \Delta_1(B_{1,i}), \phi_2 \leq \Delta_2(B_{2,j})\} : \phi \leq \phi_1 \lor \phi_2\}
= \lor\{B_{1,i} \land B_{2,j} \neq \bot : \phi_1 \leq \Delta_1(B_{1,i}), \phi_2 \leq \Delta_2(B_{2,j}), \phi \leq \phi_1 \lor \phi_2\}.
\]

But \(\phi \leq \phi_1 \lor \phi_2\), \(\phi_1 \leq \Delta_1(B_{1,i})\) and \(\phi_2 \leq \Delta_2(B_{2,j})\) if, and only if, \(\phi \leq \Delta_1(B_{1,i}) \lor \Delta_2(B_{2,j})\). So we conclude that

\[
(\rho_{\Delta_1} \lor \rho_{\Delta_2})(\phi)
= \lor\{B_{1,i} \land B_{2,j} \neq \bot : \phi \leq \Delta_1(B_{1,i}) \lor \Delta_2(B_{2,j})\}
= \lor\{B_{1,i} \land B_{2,j} \neq \bot : \phi \leq (\Delta_1 \lor \Delta_2)(B_{1,i} \land B_{2,j})\} \tag{9.37}
= \rho_{\Delta_1 \lor \Delta_2}(\phi).
\]

(2) Assume that \(\Delta\) is defined on the partition \(B_1, \ldots, B_n\) of \(\mathcal{B}\). Then \(x(\Delta)\) is also defined on \(B_1, \ldots, B_n\). The associative law of complete Boolean algebra gives us then,

\[
x(\rho_{\Delta})(\phi)
= \lor\{\rho_{\Delta}(\psi) : \phi \leq \psi = x(\psi)\}
= \lor\{\lor\{B_i : \psi \leq \Delta(B_i)\} : \phi \leq \psi = x(\psi)\}
= \lor\{B_i : \phi \leq \psi = x(\psi) \leq \Delta(B_i)\}.
\]

But, \(\phi \leq \psi = x(\psi) \leq \Delta(B_i)\) holds if, and only if, \(\phi \leq x(\Delta(B_i)) = x(\Delta)(B_i)\). Hence we see that

\[
x(\rho_{\Delta})(\phi) = \lor\{B_i : \phi \leq x(\Delta)(B_i)\} = \rho_{x(\Delta)}(\phi).
\]
This completes the proof.

As far as allocations of probability induced by simple random variables are concerned, this theorem shows that the combination and focusing of allocations reflects correctly the corresponding operations of the underlying random variables. Let \( A \) be the image of \( \mathcal{R} \), under the mapping \( \rho \). That is, \( A \) is the set of all allocations of probability which are induced by simple random variables in \( (\mathcal{B}, \mu) \). The mapping \( \Delta \mapsto \rho_\Delta \) is a homomorphism, since

\[
\rho_{\Delta_1 \vee \Delta_2} = \rho_{\Delta_1} \vee \rho_{\Delta_2},
\]
\[
\rho_x(\Delta) = x(\rho_\Delta).
\]

Also the vacuous random variable \( E \) maps to the vacuous allocation \( \nu \) and \( Z \) to \( \zeta \). Thus we conclude that the map \( \Delta \mapsto \rho_\Delta \) is a \( 0,1 \)-homomorphism between \( (\mathcal{R}, \mathcal{D}) \) and \( (\mathcal{A}_\Phi, \mathcal{D}) \) and that \( (\mathcal{A}_s, \mathcal{D}) \) is a subalgebra of the information algebra \( (\mathcal{A}_\Phi, \mathcal{D}) \). We remark that, if we restrict the mapping \( \rho \) to canonical random variables, then the mapping \( \Delta \mapsto \rho_\Delta \) becomes an embedding.

Now we turn to generalized random variables \( \Gamma \). We remind that they can be identified with random mappings into the ideal completion \( (I_\Phi, \mathcal{D}) \) of the information algebra \( (\Phi, \mathcal{D}) \) (see Section 9.5) and as such their allocation of probability is defined by \( \rho_\Gamma(\phi) = \rho_0(s_\Gamma(\phi)) \) or \( \rho_\Gamma = \rho_0 \circ s_\Gamma \) (see Section 9.4). We remark that this covers also the important case of compact information algebras \( (\Phi, \mathcal{D}) \), where the simple random variables have finite values in \( \Phi_f \), if \( (\Phi_f, \mathcal{D}) \) is a subalgebra of \( (\Phi, \mathcal{D}) \). Now, we show that the a.o.p of a generalized random variable can also be obtained as the limit of the a.o.p of the simple random variables it dominates.

**Theorem 9.13** For all \( \phi \in \Phi \) and for all generalized random variables \( \Gamma \in \mathcal{R} \), it holds that

\[
\rho_\Gamma = \bigvee \{ \rho_\Delta : \Delta \leq \Gamma \}. \tag{9.38}
\]

**Proof.** Fix an element \( \phi \in \Phi \) and consider a measurable subset \( A \subseteq s_\Gamma(\phi) \). We define a simple random variable

\[
\Delta(\omega) = \begin{cases} 
\phi & \text{if } \omega \in A, \\
\epsilon & \text{otherwise}.
\end{cases}
\]

Then certainly \( \Delta(\omega) \leq \Gamma(\omega) \) for all \( \omega \in \Omega \), hence \( \Delta \leq \Gamma \). Furthermore we have \( \rho_\Delta(\phi) = [A] \). This implies that

\[
\vee \{ \rho_\Delta(\phi) : \Delta \leq \Gamma \} \geq \vee \{ [A] : A \subseteq s_\Gamma(\phi), A \in \mathcal{A} \} = \rho_0(s_\Gamma(\phi)).
\]

Conversely, for all \( \Delta \leq \Gamma \) it holds that \( s_\Delta(\phi) \subseteq s_\Gamma(\phi) \) and that \( s_\Delta(\phi) \in \mathcal{A} \). Therefore, we conclude that

\[
\vee \{ \rho_\Delta(\phi) : \Delta \leq \Gamma \} \leq \vee \{ [A] : A \subseteq s_\Gamma(\phi), A \in \mathcal{A} \} = \rho_0(s_\Gamma(\phi)).
\]
This proves that \( \rho_\Gamma(\phi) = \bigvee \{ \rho_\Delta(\phi) : \Delta \leq \Gamma \} \) for all \( \phi \in \Phi \), hence (9.38) holds.

Theorem 9.13 shows that the a.o.p. of a generalized random variable is in the ideal completion of the information algebra \((A_s, D)\) of simple a.o.p. This ideal completion contains allocations of probability \( \rho_\Gamma : B \rightarrow \Phi \) of the random mapping associated with generalized random variables. The ideal completion of \((A_s, D)\) is a compact information algebra and (9.38) shows that the mapping \( \Gamma \mapsto \rho_\Gamma \) is continuous. It is in fact a homomorphism between the algebra of generalized random variables and their a.o.p. as the following theorem shows:

**Theorem 9.14** Let \( \Gamma, \Gamma_1, \Gamma_2 \) be generalized random variables and \( x \in D \). Then

\[
\rho_{\Gamma_1 \vee \Gamma_2} = \rho_{\Gamma_1} \vee \rho_{\Gamma_2}, \\
\rho_{x(\Gamma)}(\phi) = x(\rho_\Gamma(\phi)).
\]

The operations on the left hand side of these identities relate to the algebra of generalized random variables, whereas those on the right hand side to the algebra of a.o.p.

**Proof.** We have to show that

\[
\rho_{\Gamma_1 \vee \Gamma_2}(\phi) = (\rho_{\Gamma_1} \vee \rho_{\Gamma_2})(\phi), \\
\rho_{x(\Gamma)}(\phi) = x(\rho_\Gamma(\phi)),
\]

for all \( \phi \in \Phi \).

(1) We noted above that the mapping \( \Gamma \mapsto \rho_\Gamma \) is continuous. Therefore, using (9.20) and Lemma 6.12,

\[
\rho_{\Gamma_1 \vee \Gamma_2} = \rho_{\bigvee \{ \rho_\Delta_1 \vee \rho_\Delta_2 : \Delta_1 \leq \Gamma_1, \Delta_2 \leq \Gamma_2 \}} = \bigvee \{ \rho_\Delta_1 \vee \rho_\Delta_2 : \Delta_1 \leq \Gamma_1, \Delta_2 \leq \Gamma_2 \}.
\]

On the other hand, for every \( \phi \in I(\Phi) \), we obtain, using the associative and distributive laws of Boolean algebra,

\[
(\rho_{\Gamma_1} \vee \rho_{\Gamma_2})(\phi) = \bigvee \{ \rho_{\Gamma_1}(\phi_1) \wedge \rho_{\Gamma_2}(\phi_2) : \phi \leq \phi_1 \vee \phi_2 \} = \bigvee \{ (\rho_{\Delta_1}(\phi_1) \wedge \rho_{\Delta_2}(\phi_2)) : \Delta_1 \leq \Gamma_1, \Delta_2 \leq \Gamma_2 \}.
\]

The last equality follows from Theorem 9.12 (9.35). This proves that \( \rho_{\Gamma_1 \vee \Gamma_2} = \rho_{\Gamma_1} \vee \rho_{\Gamma_2} \).
(2) Again by continuity, we obtain from (9.21),
\[ \rho_x(\Gamma) = \rho_v(x(\Delta) : \Delta \leq \Gamma) = \vee\{\rho(\Delta) : \Delta \leq \Gamma\}. \]
But, we have also, by Theorem 9.12 (9.36) and Theorem 9.13,
\[ x(\rho(\Gamma)) = \vee\{\rho(\psi) : \psi \leq \rho(\phi) = x(\phi)\} = \vee\{\vee\{\rho(\Delta) : \Delta \leq \Gamma\} : \phi \leq \psi \leq \rho(\phi)\} = \vee\{\rho(\phi) : \phi \leq \rho(\psi) = x(\psi)\} : \Delta \leq \Gamma\} = \vee\{\rho(\Delta) : \Delta \leq \Gamma\}. \]
This proves that \( \rho_x(\Gamma) = x(\rho(\Gamma)) \).
\[ \square \]

From this theorem we conclude that that the algebra of a.o.p. of generalized random variables is the ideal completion of the information algebra of the a.o.p. simple random variables. The a.o.p. of simple random variables are the finite elements of the compact algebra of a.o.p. of generalized random variables. An a.o.p. of a generalized random variable can be approximated by the a.o.p. of simple random variables. As the generalized random variables can also be identified with random mappings, the algebra of its a.o.p. can also be seen as a subalgebra of the information algebra \((A_\Phi, D)\) of a.o.p. in the information algebra \((\Phi, D)\).

The following is a remarkable property of generalized random variables, which we formulate in the framework of compact information algebras. The interest of this theorem will become clear later especially in relation to support function, see Section ??.

**Theorem 9.15** Let \((\Phi, D)\) be a compact information algebra with finite element \(\Phi_f\) and \((\Phi_f, D)\) a subalgebra of \((\Phi, D)\). Let \(\Gamma\) be a generalized random variable in \((\Phi, D)\). Then, for any directed set \(X \subseteq \Phi\),
\[ \rho(\Gamma)(\vee X) = \bigwedge_{\phi \in X} \rho(\Gamma)(\phi) \] (9.39)

**Proof.** We prove first the identity
\[ \rho(\Delta)(\phi) = \bigwedge\{\rho(\Delta)(\psi) : \psi \in \Phi_f, \psi \leq \phi\}. \] (9.40)
for simple random variables \(\Delta\). Using the convention introduced at the beginning of the section, we write \(\Delta([B_i]) = \phi_i\), where \([B_i]\) form a partition of \(B\) for \(i = 1, \ldots, n\). Then its a.o.p. is given by \(\rho(\Delta)(\phi) = \vee\{[B_i] : \phi \leq \phi_i\}\) (see (9.34). Using this, we obtain
\[ \bigwedge\{\rho(\psi) : \psi \in \Phi_f, \psi \leq \phi\} = \bigwedge\{\vee_{\psi \leq \phi}[B_i] : \psi \in \Phi_f, \psi \leq \phi\} \]
Since the partition \([B_i]\) of \(B\) is finite, the join on the right hand side extends for every \(\psi\) only over a finite number of elements \([B_i]\). Further,
as \( \psi \) increases, the number of these elements can only decrease. But in
\[
\rho_\Delta(\phi) = \vee\{ [B_i] : \phi \leq \phi_i \}
\]
also only a finite number of elements \([B_i]\) appear and this number must be less or equal to the number of any \( \psi \leq \phi \).

So, as \( \psi \) increases towards \( \phi \), a minimal number of elements must be attained for some \( \psi_0 \leq \phi \). Say this number is \( m \) and assume that the elements are numbered as \([B_1], \ldots, [B_m]\). Then we have that the infimum
\[
\wedge\{ \rho_\Gamma(\psi) : \psi \in \Phi_f, \psi \leq \phi \} = \vee\{ [B_i] : \phi \leq \phi_i \}
\]
and this proves (9.40).

Next, we extend (9.40) to any generalized random variables \( \Gamma = \vee\{ \Delta : \Delta \in \mathcal{R}, \Delta \leq \Gamma \} \). For this purpose we use the distributive law in complete Boolean algebra \( B \):
\[
\rho_\Gamma(\phi)
= \vee\{ \rho_\Gamma(\psi) : \psi \in \Phi_f, \psi \leq \phi \} : \Delta \in \mathcal{R}, \Delta \leq \Gamma
= \wedge\{ \rho_\Gamma(\psi) : \Delta \in \mathcal{R}, \Delta \leq \Gamma \} : \psi \in \Phi_f, \psi \leq \phi
= \wedge\{ \rho_\Gamma(\psi) : \psi \in \Phi_f, \psi \leq \phi \}
\]
(9.41)

To conclude, let \( X \subseteq \Phi \) be directed. Consider \( \psi \in X \). Then \( \psi \leq \vee X \), hence \( \rho_\Gamma(\psi) \geq \rho_\Gamma(\vee X) \), and it follows that \( \wedge_{\psi \in X} \rho_\Gamma(\psi) \geq \rho_\Gamma(\vee X) \). Further, if \( \eta \) is a finite element and \( \eta \leq \vee X \), then there is a \( \psi \in X \) such that \( \eta \leq X \). This implies that \( \rho_\Gamma(\psi) \geq \rho_\Gamma(\eta) \). From this we conclude, using (9.41)
\[
\rho_\Gamma(\vee X)
= \wedge\{ \rho_\Gamma(\psi) : \psi \in \Phi_f, \psi \leq \phi \}
\geq \wedge_{\psi \in X} \rho_\Gamma(\psi).
\]

This proves finally (9.39). \( \square \)

Following (Shafer, 1979) we call an allocation of probability satisfying (9.39) condensable. Thus, the a.o.p.s associated with generalized random variables are condensable. The significance of this condition becomes more evident in Section ??.

Next we examine the case of random variables and their allocations of probability. We look at them in the context of a \( D \)-compact information algebra \( (\Phi, D) \) with finite elements \( \Phi_f \) and assume that \( (\Phi_f, D) \) is a subalgebra of \( (\Phi, D) \). According to the Representation Theorem 6.7 this is like looking at an information algebra \( (\Phi, D) \) and considering its ideal completion \( (I_\Phi, D) \). We remind that \( (\sigma(\Phi_f), D) \) is a \( \sigma \)-information algebra, a subalgebra of \( (\Phi, D) \).

A random variable \( \Gamma \) is the join (or the limit) of a monotone nondecreasing sequence of simple random variables \( \Delta_i \) with \( \Delta_1 \leq \Delta_2 \leq \ldots \)
9.8. RANDOM MAPPINGS AND ALLOCATIONS

\[ \Gamma = \bigvee_{i=1}^{\infty} \Delta_i. \]  
The simple random variables take values in \( \Phi_f \), and the random variable \( \Gamma \) in \( \sigma(\Phi) \). By Lemma 9.6 a random variable \( \Gamma \) is a also a generalized random variable. Therefore Theorem 9.14 applies also to random variables. So, the mapping \( \Gamma \mapsto \rho_\Gamma \) is a homomorphism of the information algebra \((\mathcal{R}_\sigma, D)\) of random variables into the information algebra \((\mathcal{A}_\Phi, D)\) of a.o.p.s.

We are going to show more, namely that the map \( \Gamma \mapsto \rho_\Gamma \) is a \( \sigma \)-homomorphism from the \( \sigma \)-information algebra \((\mathcal{R}_\sigma, D)\) into the information algebra \((\mathcal{A}_\Phi, D)\).

**Theorem 9.16** Suppose \((\Phi, D)\) to be a \( \sigma \)-information algebra, and \( \Gamma_i \in \mathcal{R}_\sigma \) for \( i = 1, 2, \ldots \). Then

\[
\rho_{\bigvee_{i=1}^{\infty} \Gamma_i} = \bigvee_{i=1}^{\infty} \rho_{\Gamma_i}. \tag{9.42}
\]

**Proof.** Since the mapping \( \Gamma \mapsto \rho_\Gamma \) is a homomorphism, it maintains order. As a random variable, \( \Gamma \) equals \( \bigvee_{i=1}^{\infty} \Delta_i \), where \( \Delta_i \) form a monotone sequence of simple random variables. Since \( \Gamma \) is also a generalized random variable, we have by (9.41) \( \rho_\Gamma = \bigvee \{ \rho_\Delta(\phi) : \rho_\Delta \in \mathcal{R}_\sigma, \Delta \leq \Gamma \} \). The monotone sequence \( \Delta_i \) is directed in \( \mathcal{R} \). By compactness there is for every \( \Delta \leq \Gamma \) an index \( j \) so that \( \Delta \leq \Delta_j \). This implies \( \rho_\Delta \leq \rho_{\Delta_j} \) from which it follows that \( \rho_\Gamma \leq \bigvee_{i=1}^{\infty} \rho_{\Delta_i} \). The converse inequality is evident. So we conclude that

\[
\rho_\Gamma = \bigvee_{i=1}^{\infty} \rho_{\Delta_i}, \tag{9.43}
\]

if \( \Gamma = \bigvee_{i=1}^{\infty} \Delta_i \).

Consider now the generalized random variables \( \Gamma_i \) for \( i = 1, 2, \ldots \) and \( \Gamma = \bigvee_{i=1}^{\infty} \Gamma_i \). Let \( \Gamma_i = \bigvee_{j=1}^{\infty} \Delta_{i,j} \), where for every \( i = 1, 2, \ldots \) the \( \Delta_{i,1}, \Delta_{i,2}, \ldots \) is a monotone sequence of simple random variables. Then

\[
\Gamma = \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} \Delta_{i,j}.
\]

In the standard way, we define \( \Delta_i = \bigvee_{h=1}^{i} \bigvee_{j=1}^{h} \Delta_{h,j} \). The \( \Delta_i \) form a monotone sequence of simple random variables and \( \Gamma = \bigvee_{i=1}^{\infty} \Delta_i \). By (9.43), and the homomorphism between simple random variables and their a.o.p.s we obtain

\[
\rho_\Gamma = \bigvee_{i=1}^{\infty} \rho_{\Delta_i} = \bigvee_{i=1}^{\infty} \left( \bigvee_{h=1}^{i} \bigvee_{j=1}^{h} \rho_{\Delta_{i,j}} \right) = \bigvee_{i=1}^{\infty} \left( \bigvee_{j=1}^{\infty} \rho_{\Delta_{i,j}} \right) = \bigvee_{i=1}^{\infty} \rho_{\Gamma_i}.
\]

This proves (9.42). \( \square \)

As a preparation for an interpretation of this result, we remark that for a \( \sigma \)-information algebra the following general result holds:
Lemma 9.7 Suppose \((\Phi, D)\) to be a \(\sigma\)-information algebra, \(\Gamma\) a random mapping. Then

\[
s_{\Gamma}(\bigvee_{i=1}^{\infty} \phi_i) = \bigcap_{i=1}^{\infty} s_{\Gamma}(\phi_i). \tag{9.44}
\]

Proof. We have

\[
s_{\Gamma}(\bigvee_{i=1}^{\infty} \phi_i) = \{\omega \in \Omega : \bigvee_{i=1}^{\infty} \phi_i \leq \Gamma(\omega)\}.
\]

Let \(\phi = \bigvee_{i=1}^{\infty} \phi_i\). Since \(\phi_i \leq \phi\) we conclude that \(s_{\Gamma}(\phi) \subseteq s_{\Gamma}(\phi_i)\), hence \(s_{\Gamma}(\phi) \subseteq \bigcap_{i=1}^{\infty} s_{\Gamma}(\phi_i)\). On the other hand, consider \(\omega \in \bigcap_{i=1}^{\infty} s_{\Gamma}(\phi_i)\), that is \(\phi_i \leq \Gamma(\omega)\) for all \(i = 1, 2, \ldots\). Then we have \(\bigvee_{i=1}^{\infty} \phi_i = \phi \leq \Gamma(\omega)\), hence \(\omega \in s_{\Gamma}(\phi)\). This shows that \(s_{\Gamma}(\phi) \supseteq \bigcap_{i=1}^{\infty} s_{\Gamma}(\phi_i)\) and this proves (9.44). \(\Box\)

Since for any random variable \(\Gamma\) and every \(\phi \in \Phi\), we have \(\rho_{\Gamma}(\phi) = \rho_0(s_{\Gamma}(\phi))\) and the mapping \(\rho_0\) is a \(\sigma\)-homomorphism from the power set of \(\Omega\) onto \(B\) (see Theorem 9.2) it follows also from (9.44)

\[
\rho_{\Gamma}(\bigvee_{i=1}^{\infty} \phi_i) = \bigwedge_{i=1}^{\infty} \rho_{\Gamma}(\phi_i).
\]

An allocation of probability, which satisfies this identity is called a \(\sigma\)-allocation of probability. Thus, a random variables induces a \(\sigma\)-a.o.p. Let \(A_{\sigma}\) denote the image of \(R_{\sigma}\) under the mapping \(\Gamma \mapsto \rho_{\Gamma}\) in \(A_{\Phi}\).

Next we show that continuity of extraction is also satisfied in the algebra \((A_{\sigma}, D)\):

Theorem 9.17 Let \((\Phi, D)\) to be a compact information algebra, and \(\Gamma_i \in R_{\sigma}\) for \(i = 1, 2, \ldots\) a monotone sequence, \(\Gamma_1 \leq \Gamma_2 \leq \ldots\). Then for every \(x \in D\),

\[
x(\bigvee_{i=1}^{\infty} \rho_{\Gamma_i}) = \bigvee_{i=1}^{\infty} x(\rho_{\Gamma_i}). \tag{9.45}
\]

Proof. The proof bases on the continuity of extraction in the \(\sigma\)-information algebra \((R_{\sigma}, D)\) of random variables, Theorem 9.5

\[
x(\bigvee_{i=1}^{\infty} \Gamma_i) = \bigvee_{i=1}^{\infty} x(\Gamma_i).
\]

Take the a.o.p. of both sides. On the left this leads to, using the fact that the mapping is a homomorphism of generalized random variables, Theorem 9.13, and Theorem 9.16

\[
\rho_x(\bigvee_{i=1}^{\infty} \Gamma_i) = x(\rho_{\bigvee_{i=1}^{\infty} \Gamma_i}) = x(\bigvee_{i=1}^{\infty} \rho_{\Gamma_i}).
\]
On the right hand side we obtain by the same argument

\[ \rho_{\bigvee_{i=1}^{\infty} x(\Gamma_i)} = \bigvee_{i=1}^{\infty} \rho_x(\Gamma_i) = \bigvee_{i=1}^{\infty} x(\rho_{\Gamma_i}) \]

This proves the identity (9.45).

What can be said about the mapping \( \Gamma \mapsto \rho_{\Gamma} \) for random mappings \( \Gamma \) in general? Let \( (\Phi, D) \) be an information algebra, \( (\Omega, A, P) \) a probability space and \( \Gamma : \Omega \to \Phi \) a random mapping. The mapping \( \Gamma \mapsto \rho_{\Gamma} \) is obviously order-preserving: \( \Gamma_1 \leq \Gamma_2 \) means that \( \Gamma_1(\omega) \leq \Gamma_2(\omega) \) for all \( \omega \in \Omega \). This implies that \( s_{\Gamma_1}(\phi) \subseteq s_{\Gamma_2}(\phi) \) for all \( \phi \in \Phi \), and from this it follows that \( \rho_{\Gamma_1}(\phi) = \rho_0(s_{\Gamma_1}(\phi)) \leq \rho_0(s_{\Gamma_2}(\phi)) = \rho_{\Gamma_1}(\phi) \) for all \( \phi \in \Phi \), hence \( \rho_{\Gamma_1} \leq \rho_{\Gamma_2} \).

But the mapping is no more a homomorphism. In fact, let \( \Gamma_1 \) and \( \Gamma_2 \) be two random mappings. Then the support of the combination of these random mappings is

\[
s_{\bigvee \Gamma_1 \Gamma_2}(\phi) = \{ \omega : \phi \leq \bigvee \Gamma_1(\omega) \vee \Gamma_2(\omega) \} = \bigcup \{ \omega : \phi \leq \bigvee \Gamma_1(\omega), \phi_2 \leq \bigvee \Gamma_2(\omega), \phi \leq \phi_1 \vee \phi_2 \} = \bigcup \{ s_{\Gamma_1}(\phi_1) \cap s_{\Gamma_2}(\phi_2) : \phi \leq \phi_1 \vee \phi_2 \}.
\]

Note that

\[
\rho_0 \left( \bigcup_i H_i \right) = \bigvee \{ [A] : A \in A, A \subseteq \bigcup_i H_i \} \\
\geq \bigvee_i \{ [A] : A \in A, A \subseteq H_i \} = \bigvee_i \rho_0(H_i).
\]

This implies then for all \( \phi \in \Phi \)

\[
\rho_{\bigvee \Gamma_1 \Gamma_2}(\phi) = \rho_0 \left( \bigcup \{ s_{\Gamma_1}(\phi_1) \cap s_{\Gamma_2}(\phi_2) : \phi \leq \phi_1 \vee \phi_2 \} \right) \geq \bigvee \{ \rho_0(s_{\Gamma_1}(\phi_1) \cap s_{\Gamma_2}(\phi_2)) : \phi \leq \phi_1 \vee \phi_2 \} = \bigvee \{ \rho_{\Gamma_1}(\phi_1) \wedge \rho_{\Gamma_2}(\phi_2) : \phi \leq \phi_1 \vee \phi_2 \} = (\rho_{\Gamma_1} \vee \rho_{\Gamma_2})(\phi).
\]

(9.46)

So, we have \( \rho_{\bigvee \Gamma_1 \Gamma_2} \geq \rho_{\Gamma_1} \vee \rho_{\Gamma_2} \). Equality holds only in particular cases, like for instance for generalized or \( \sigma \)-random variables. Since \( \rho_{\bigvee \Gamma_1 \Gamma_2} \) allocates more probability to a hypothesis \( \phi \in \Phi \) than \( \rho_{\bigvee \Gamma_1 \Gamma_2} \) does, it seems that by the map to the allocation of probability some information is lost in general.
Consider also extraction, that is a random mapping \( \Gamma \) and \( x \in D \). Then, since \( (x(\Gamma))(\omega) = x(\Gamma(\omega)) \),

\[
\begin{align*}
s_{x(\Gamma)}(\phi) & = \{ \omega \in \Omega : \phi \leq x(\Gamma(\omega)) \} \\
& = \bigcup \{s_{\Gamma}(\psi) : \psi = x(\psi), \phi \leq \psi \}.
\end{align*}
\] (9.47)

Thus, we obtain for the a.o.p. of \( x(\Gamma) \),

\[
\begin{align*}
\rho_{x(\Gamma)}(\phi) & = \rho_0 \bigcup \{s_{\Gamma}(\psi) : \psi = x(\psi), \phi \leq \psi \} \\
& \geq \bigvee \{\rho_0(s_{\Gamma}(\psi)) : \psi = x(\psi), \phi \leq \psi \} \\
& \geq \bigvee \{\rho_{\Gamma}(\psi) : \psi = x(\psi), \phi \leq \psi \} \\
& = (x(\rho_{\Gamma}))(\phi).
\end{align*}
\]

So, here we find that \( \rho_{x(\Gamma)} \geq x(\rho_{\Gamma}) \) and again equality holds only in particular cases. This is another indication that the random mapping \( \Gamma \) contains more information than its a.o.p. \( \rho_{\Gamma} \).
Chapter 10

Support Functions

10.1 Characterization

As we have noted in Chapter 9, we may consider a random mapping \( \Gamma \) as an information available to us: \( \Gamma(\omega) \) is a “piece of information”, which can be asserted, provided \( \omega \) is the sample element chosen by a chance process, or the “correct”, but unknown assumption in a set of possible assumptions \( \Omega \). Here, information \( \Gamma(\omega) \) may either be an element of the set \( \Phi \) of an information algebra \( (\Phi, D) \) or else an ideal of \( \Phi \), hence an element of the ideal completion \( (I\Phi, D) \) of \( (\Phi, D) \). We have defined the allocation of support \( s_{\Gamma}(\phi) \) of a random mapping to an element \( \phi \in \Phi \) as the set of elements \( \omega \in \Omega \), which imply \( \phi \), i.e. such that \( \phi \in \Gamma(\omega) \) or \( \phi \leq \Gamma(\omega) \), see Sections 9.3 and 9.4. Any \( \omega \in s_{\Gamma}(\phi) \) is an assumption, i.e. an argument, which permits to infer the piece of information \( \phi \) in the light of the random mapping \( \Gamma \). So the larger the set \( s_{\Gamma}(\phi) \), the more arguments are present to support \( \phi \). Or, more to the point, the more probable, the more likely it is that the correct assumption \( \omega \) belongs to \( s_{\Gamma}(\phi) \), the stronger the hypothesis of information \( \phi \) is supported. This probability was denoted by \( sp_{\Gamma}(\phi) \) and called the degree of support of a hypothesis by a random mapping \( \Gamma \). We refer to Chapter 9 for this point of view. The allocation of support can be seen as a set-valued map \( s_{\Gamma} : \Phi \to \mathcal{P}(\omega) \) from the information algebra, or its ideal completion \( s_{\Gamma} : I\Phi \to \mathcal{P}(\omega) \) into the sample space. Similarly, the degrees of support can be seen as a numerical map or function \( sp_{\Gamma} : \Phi \to [0,1] \) of \( \Phi \) into the unit interval. The goal of this chapter is to study this function.

We do not exclude in this Chapter that \( \Gamma(\omega) = z \) for some \( \omega \). This represents improper information, which can be interpreted as contradictory information. Under semantic aspects such improper information could and should be excluded. We refer to Section 9.3 for a discussion of this issue in the context of simple random functions. If \( \Gamma(\omega) \neq z \) for all \( \omega \), the random mapping is called normalized.
Consider then a random mapping \( \Gamma : \Omega \to \Phi \) from a probability space \((\Omega, \mathcal{A}, P)\) into an information algebra \((\Phi, D)\). The corresponding support is defined for any \( \phi \in \Phi \) as

\[
s_{\Gamma}(\phi) = \{ \omega \in \Omega : \phi \leq \Gamma(\omega) \}.
\]

The set \( s_{\Gamma} \) thus contains all assumptions \( \omega \) for which \( \Gamma(\omega) \) implies \( \phi \). The following theorem collects elementary properties of the mapping \( s_{\Gamma} : \Phi \to \mathcal{P}(\Omega) \):

**Theorem 10.1** If \( \Gamma : \Omega \to \Phi \), then

1. \( s_{\Gamma}(0) = \Omega \),
2. If \( \phi \leq \psi \), then \( s_{\Gamma}(\psi) \subseteq s_{\Gamma}(\phi) \),
3. \( s_{\Gamma}(\phi \vee \psi) = s_{\Gamma}(\phi) \cap s_{\Gamma}(\psi) \) for all \( \phi, \psi \in \Phi \),
4. if \( \Gamma \) is normalized, then \( s_{\Gamma}(1) = \emptyset \).

**Proof.** (1) follows since \( 0 \) is the least element in \( \Phi \), hence \( 0 \leq \Gamma(\omega) \) for all \( \omega \in \Omega \). (2) is evident. (3) follows, since \( \phi, \psi \leq \Gamma(\omega) \) if and only if \( \phi \vee \psi \leq \Gamma(\omega) \) and (4) follows from the definition of a normalized random mapping. \( \Box \)

Sometimes \( \Phi \) may be a \( \sigma \)-semilattice or even a complete lattice, for instance, if \((\Phi, D)\) is a compact information algebra. Then something more can be said about the support of a random mapping.

**Theorem 10.2** Let \( \Gamma : \Omega \to \Phi \) be a random mapping.

1. If \( \Phi \) is a \( \sigma \)-semilattice, \( \phi_1, \phi_2, \ldots \in \Phi \), then

\[
s_{\Gamma}(\bigvee_{i=1}^{\infty} \phi_i) = \bigcap_{i=1}^{\infty} s_{\Gamma}(\phi_i). \tag{10.1}
\]

2. If \( \Phi \) is a complete lattice, \( X \subseteq \Phi \), then

\[
s_{\Gamma}(\bigvee X) = \bigcap_{\phi \in X} s_{\Gamma}(\phi). \tag{10.2}
\]

**Proof.** (1) We have \( \phi_1, \phi_2, \ldots \leq \Gamma(\omega) \) if and only if \( \bigvee_{i=1}^{\infty} \phi_i \leq \Gamma(\omega) \). This implies (10.1).

(2) Similarly, we have \( \phi \leq \Gamma(\omega) \) for all \( \phi \in X \) if and only if \( \bigvee X \leq \Gamma(\omega) \) and this implies (10.2). \( \Box \)

We want to make use of the probability space \((\Omega, \mathcal{A}, P)\) to judge the likelihood that a random mapping \( \Gamma \) supports a hypothesis \( \phi \in \Phi \). The degree of support \( sp_{\Gamma}(\phi) \) of an element \( \phi \in \Phi \) is measured by the probability of its support \( s_{\Gamma}(\phi) \), provided this probability is defined. But this is the case only if \( s_{\Gamma}(\phi) \in \mathcal{A} \). Therefore, we define:
10.1. CHARACTERIZATION

Definition 10.1 If $\Gamma : \Omega \rightarrow \Phi$ is a random mapping from a probability space $(\Omega, \mathcal{A}, P)$ into an information algebra $(\Phi, D)$, then $\phi \in \Phi$ is called $\Gamma$-measurable, if $s_{\Gamma}(\phi) \in \mathcal{A}$.

The set of all $\Gamma$-measurable elements $\phi \in \Phi$ is denoted by $\mathcal{E}_\Gamma$.

Theorem 10.3 For any random variable $\Gamma$, $\mathcal{E}_\Gamma$ is a subsemilattice of $\Phi$, containing $0$; if $\Gamma$ is normalized, then $1$ belongs to $\mathcal{E}_\Gamma$ too. Further, if $\Phi$ is a $\sigma$-semilattice or a complete lattice, then $\mathcal{E}_\Gamma$ is a $\sigma$-semilattice.

Proof. The first part of the theorem follows directly from the definition of $\mathcal{E}_\Gamma$ and Theorem 10.1. The second part follows from Theorem 10.2.

On the semilattice $\mathcal{E}_\Gamma$ we define $s_{\Gamma}(\phi) = P(s_{\Gamma}(\phi))$. Thus, $s_{\Gamma}$ is a function with values in $[0, 1]$, defined on $\mathcal{E}_\Gamma$. This function is called the support function of the random mapping $\Gamma$. The next theorem collects the basic properties of this function.

Theorem 10.4 Let $\Gamma$ be a random mapping from the probability space $(\Omega, \mathcal{A}, P)$ into the information algebra $(\Phi, D)$, and $s_{\Gamma}$ the associated support function, defined on $\mathcal{E}_\Gamma$. Then $s_{\Gamma}$ has the following properties:

1. $s_{\Gamma}(0) = 1$.
2. If $\phi_1, \ldots, \phi_n \geq \phi, \phi_1, \ldots, \phi_m, \phi \in \mathcal{E}_\Gamma$,

$$s_{\Gamma}(\phi) \geq \sum_{\emptyset \neq I \subseteq \{1, \ldots, m\}} (-1)^{|I|+1} s_{\Gamma}(\bigvee_{i \in I} \phi_i). \quad (10.3)$$

3. If $\Phi$ is a $\sigma$-semilattice or a complete lattice, and if $\phi_1 \leq \phi_2 \leq \ldots \in \mathcal{E}_\Gamma$,

$$s_{\Gamma}(\bigvee_{i=1}^\infty \phi_i) = \lim_{i \to \infty} s_{\Gamma}(\phi_i). \quad (10.4)$$

4. If $\Gamma$ is normalized, then $s_{\Gamma}(1) = 0$.

Proof. (1) and (4) follow from Theorem 10.1 (1) and (4).

(2) Note that by Theorem 10.1 (2) we have $s_{\Gamma}(\bigvee_{i \in I} \phi_i) = P(s_{\Gamma}(\bigvee_{i \in I} \phi_i)) = P(\bigcap_{i \in I} s_{\Gamma}(\phi_i))$ for a finite index set. On the right hand side of (10.3) we have then by the inclusion-exclusion formula of probability theory,

$$\sum_{\emptyset \neq I \subseteq \{1, \ldots, m\}} (-1)^{|I|+1} P(\bigcap_{i \in I} s_{\Gamma}(\phi_i)) = P(\bigcup_{i=1}^m s_{\Gamma}(\phi_i)).$$

But $\phi \leq \phi_1, \ldots, \phi_m$ implies $s_{\Gamma}(\phi) \supseteq s_{\Gamma}(\phi_i)$, hence

$$s_{\Gamma}(\phi) \supseteq \bigcup_{i=1}^m s_{\Gamma}(\phi_i)$$
This proves (10.3)

(3) In this case \(E\) is a \(\sigma\)-semilattice (Theorem 10.3), that is, \(\phi_1 \leq \phi_2 \leq \ldots \in E\) implies that \(\vee_{i=1}^{\infty} \phi_i \in E\). Further, by Theorem 10.2, \(sp(\vee_{i=1}^{\infty} \phi_i) = P(\cap_{i=1}^{\infty} s_{\Gamma}(\phi_i))\). Now, \(sp(\phi_1) \geq sp(\phi_2) \geq \ldots \) (Theorem 10.1 (2)). By continuity of probability it follows that \(P(\cap_{i=1}^{\infty} s_{\Gamma}(\phi_i)) = \lim_{i \to \infty} P(s_{\Gamma}(\phi_i))\). This proves (10.4).

As consequence we deduce from (2) of the theorem above that for \(\phi \leq \psi\) we have \(sp(\psi) \leq sp(\phi)\). Thus the function \(sp\) is monotone. In fact a function satisfying property (2) of the theorem above is called \textit{monotone of order} \(\infty\).

In Section 9.4 we proposed to extend the support function of a random mapping \(\Gamma\) beyond the measurable elements by \(sp(\psi) = \mu(\rho(\psi))\), where \(\rho(\phi) = \rho_0(s_{\Gamma}(\phi))\) is an allocation of probability associated with the random mapping \(\Gamma\), see (9.12) and \((\mu, B)\) is the probability algebra associated with the probability space \((\Omega, A, P)\). Any allocation of probability \(\rho : B \to \Phi\) generates a function \(sp = \mu \circ \rho\) which satisfies properties (1) and (2) of Theorem 10.4. Therefore, in particular the function \(sp = \mu \circ \rho\), which is defined on \(\Phi\), and even \(I_{\Phi}\) has the properties stated in Theorem 10.4.

\textbf{Theorem 10.5} Let \((\mu, B)\) be a probability algebra, \(\rho : \Phi \to B\) an allocation of probability, and \(sp = \mu \circ \rho\).

1. \(sp\) satisfies properties (1) and (2) of Theorem 10.4

2. If \(\Phi\) is a \(\sigma\)-semilattice and \(\rho\) a \(\sigma\)-allocation of probability, then (3) of Theorem 10.4 holds.

3. If \(\Phi\) is a complete lattice and for any directed set \(X \subseteq \Phi\)

\[ \rho(\vee X) = \wedge_{\phi \in X} \rho(\phi) \]

holds, then

\[ sp(\vee X) = \inf_{\phi \in X} sp(\phi). \] (10.5)

\textit{Proof.} (1) and (2) This is proved as in proof of Theorem 10.4.

(3) The set \(\{\rho(\phi) : \phi \in X\}\) is downwards directed if \(X\) is directed. Therefore, by Lemma 9.1

\[ \mu(\rho(\vee X)) = \mu(\wedge_{\phi \in X} \rho(\phi)) = \inf_{\phi \in X} \mu(\rho(\phi)). \]

This proves (10.5).

Next, we consider \textit{compact} information algebras \((\Phi, D)\), with finite elements \(\Phi_f\). We always suppose that \((\Phi_f, D)\) is a subalgebra of \((\Phi, D)\) so that \(\Phi\) can be considered as the ideal completion \(I_{\Phi_f}\) of \(\Phi_f\) (see Theorem
6.7). Or, in other words, the results to be derived below apply also to the ideal completion $I_{\Phi}$ of any information algebra $(\Phi, D)$. In this context we remind that a generalized random variable $\Gamma$ is the supremum of the simple random variables, it dominates, $\Gamma = \bigvee\{\Delta : \Delta \in \mathcal{R}, \Delta \leq \Gamma\}$. Simple random variables are here and in the sequel always assumed to take finite elements as values, that is $\Delta(\omega) \in \Phi_f$ for all $\omega$. In such a case, the support function of a generalized random variable can be approximated by its values for finite elements.

**Theorem 10.6** Let $(\Phi, D)$ be a compact information algebra, with $\Phi_f$ as finite elements and $\Gamma$ a generalized random variable with values in $(\Phi, D)$. Further let $\text{sp}_\Gamma(\psi) = \inf_{\phi \in \Phi_f} \psi \leq \phi$. Then for all $\phi \in \Phi$,

$$s\Gamma(\phi) = \inf\{s\Gamma(\psi) : \psi \in \Phi_f, \psi \leq \phi\}. \quad (10.6)$$

Furthermore, if $X \subseteq \Phi$ is directed, then

$$s\Gamma(\bigvee X) = \inf_{\phi \in X} s\Gamma(\phi). \quad (10.7)$$

**Proof.** We are going to prove (10.7), (10.7) is a particular case of (10.7). By Theorem 9.15 we have $\rho(\bigvee X) = \wedge_{\phi \in X} \rho(\phi)$. Then (10.7) follows from Theorem 10.5 (3).

In the same framework, if $\Gamma = \bigvee_{i=1}^\infty \Delta_i$ is a random variable defined by a sequence family of simple random variables $\Delta_1, \Delta_2, \ldots$, then the degree of support of any element in $\sigma(\Phi_f)$ may be obtained as a limit of the degrees of support of finite elements. In fact, if $\phi \in \sigma(\Phi_f)$, then $\phi = \bigvee_{i=1}^\infty \psi_i$, where $\psi_i \in \Phi_f$ (Theorem 9.3). We may always assume that the sequence $\psi_i$ is monotone, $\psi_1 \leq \psi_2 \leq \ldots$. Then this sequence is a directed set in $\Phi$ and Theorem 10.6 applies. But due to the monotonicity of the sequence, we have

$$\inf\{s\Gamma(\psi_i) : i = 1, 2, \ldots\} = \lim_{i \to \infty} s\Gamma(\psi_i).$$

So, if $\phi = \bigvee_{i=1}^\infty \psi_i$, then

$$s\Gamma(\phi) = \lim_{i \to \infty} s\Gamma(\psi_i). \quad (10.8)$$

The degree of support of a random variable can in some cases also be approximated by the degrees of support of the simple random variables which approximate the random variable.

**Theorem 10.7** Let $(\Phi, D)$ be an information algebra and $(\sigma(\Phi), D)$ its $\sigma$-extension in $I_{\Phi}$. If $\Gamma = \bigvee_{i=1}^\infty \Delta_i$, where $\Delta_i$ are simple random variables with values in $\Phi$, is a random variable defined on the probability space $(\Omega, \mathcal{A}, P)$ with values in $(\sigma(\Phi), D)$, then all elements $\phi \in \Phi$ are $\Gamma$-measurable, $\mathcal{E}_\Gamma = \Phi$. Furthermore, if the $\Delta_i$ form a monotone increasing sequence of simple random variables, then for all $\phi \in \Phi$,

$$s\Gamma(\phi) = \lim_{i \to \infty} s\Delta_i(\phi). \quad (10.9)$$
Proof. If $\Gamma$ is a random variable, defined by $\Gamma = \lor_{i=1}^{\infty} \Delta_i$, we may always assume that the $\Delta_i$ form monotone sequence of simple random variables. Consider any $\phi \in \Phi$ and its support $s_{\Gamma}(\phi)$ relative to the random variable $\Gamma$. Then $\Delta_i \leq \Gamma$ implies $s_{\Delta_i}(\phi) \subseteq s_{\Gamma}(\phi)$, hence $\bigcup_{i=1}^{\infty} s_{\Delta_i}(\phi) \subseteq s_{\Gamma}(\phi)$. On the other hand

$$s_{\Gamma}(\phi) = \{\omega \in \Omega : \phi \leq \lor_{i=1}^{\infty} \Delta_i(\omega)\}.$$ 

Consider an $\omega \in s_{\Gamma}(\phi)$. As a monotone sequence, the $\Delta_i(\omega)$ form a directed set. Its supremum $\Gamma(\omega)$ belongs to the compact information algebra $(I_\Phi, D)$, where the elements of $\Phi$ are the finite elements. Therefore, by compactness, there must be an index $i$ such that $\phi \leq \Delta_i(\phi)$, hence $\omega \in s_{\Delta_i}(\phi)$. But this shows that $s_{\Gamma}(\phi) \subseteq \bigcup_{i=1}^{\infty} s_{\Delta_i}(\phi)$, hence

$$s_{\Gamma}(\phi) = \bigcup_{i=1}^{\infty} s_{\Delta_i}(\phi). \quad (10.10)$$

Now, $s_{\Delta_i}(\phi)$ is measurable for all $i$, hence $s_{\Gamma}(\phi)$ is so too. This proves the first part of the theorem.

If the sequence of the $\Delta_i$ is monotone increasing, then so is $s_{\Delta_i}(\phi)$ for any $\phi \in \Phi$. Then (10.9) follows from (10.10) and the continuity of probability.

We are in this chapter going to study functions satisfying properties (1) and (2) from the Theorem 10.4 above. As we have seen, such functions do arise from random mappings in different ways and also from allocations of probability. Therefore, we are going to define a corresponding class of functions.

**Definition 10.2** Let $E$ be a join-semilattice with a least element 0. Then a function $sp : E \to [0,1]$ satisfying (1) and (2) below is called a support function on $E$:

1. $sp(0) = 1$.

2. If $\phi_1, \ldots, \phi_n \geq \phi, \phi_1, \ldots, \phi_m, \phi \in E$,

$$sp(\phi) \geq \sum_{\emptyset \neq I \subseteq \{1, \ldots, m\}} (-1)^{|I|+1} sp(\lor_{i \in I} \phi_i). \quad (10.11)$$

3. If in addition $E$ is closed under countable joins, and for any monotone sequence $\phi_1 \leq \phi_2 \leq \cdots$ the condition

$$sp(\lor_{i=1}^{\infty} \phi_i) = \lim_{i \to \infty} sp(\phi_i) \quad (10.12)$$

holds, then $sp$ is called a continuous support function of $E$. 

4. If further $\mathcal{E}$ is a complete semi lattice and for any directed set $X \subseteq \mathcal{E}$,

$$sp(\lor X) = \inf_{\phi \in X} sp(\phi) \quad (10.13)$$

holds, then $sp$ is called a condensable support function on $\mathcal{E}$.

So, for any random mapping $\Gamma$, the function $sp_{\Gamma}$ is a support function on $\mathcal{E}_{\Gamma}$. Random variables $\Gamma$ have continuous support functions $sp_{\Gamma}$ and the support functions $sp_{\Gamma}$ of generalized random variables $\Gamma$ are condensable on $\Phi$, if $(\Phi, D)$ is a compact information algebra. A function, which satisfies (10.11) of the definition above is also called monotone of order $\infty$. (Choquet, 1953–1954; Choquet, 1969). We are, in the sequel, going to study such support functions. The first question is whether any support function can be obtained as the support function of a random mapping. This question will be addressed in the next section. Further, if a support function is defined on some sub-semilattice $\mathcal{E}$ of an information algebra $(\Phi, D)$, how can this function be extended to a ll of $\Phi$? This question will be studied in Section 10.3.

### 10.2 Generating Support Functions

Any random mapping $\Gamma$ from some probability space $(\Omega, \mathcal{A}, P)$ into an information algebra $(\Phi, D)$ generates a support function $sp_{\Gamma}$ on the join-semilattice $\mathcal{E}_{\Gamma} \subseteq \Phi$ of its $\Gamma$-measurable elements. We remind that $\mathcal{E}_{\Gamma}$ contains at least the element $0$ of $\Phi$. Now, suppose that $\mathcal{E}$ is a join-semilattice containing a least element $0$ and that $sp : \mathcal{E} \to \mathbb{R}$ is a support function according to Definition 10.2 in the previous section. In fact, we shall always consider $\mathcal{E}$ as a sub-semilattice of some information algebra $(\Phi, D)$. Is there a random mapping $\Gamma$ into $\Phi$ such that its support function $sp_{\Gamma}$ concides with $sp$ on $\mathcal{E}$? We show in this section that the answer is affirmative, with the small amendment, that the mapping is into the ideal completion $I_0$ of $\Phi$ rather than into $\Phi$.

This result is based on the Theorem of Krein-Milman which states that in a locally convex topological space, which is Hausdorff, any compact convex set $S$ is the closure of the convex hull of its extreme points (Phelps, 2001). The set $S$ will be formed by the support functions in the space of real-valued functions on $\mathcal{E}$. We shall use a result of Choquet on the extreme points of monotone functions of order $\infty$ (Choquet, 1953–1954). In fact, the theory presented here can be seen as part of Choquet’s theory of capacities, and illustrates in particular the connection of capacities to probability.

Let $\mathcal{E}$ be a semi lattice, containing the least element $0$. Consider the vector space of functions $f : \mathcal{E} \to \mathbb{R}$ with pointwise addition and scalar multiplication. It becomes a topological space with pointwise convergence. Since $\mathbb{R}$ is Hausdorff, so is $V$ (Kelley, 1955). Define $p_\phi(f) = |f(\phi)|$ for $f \in V$ and $\phi \in \mathcal{E}$. Then $p_\phi$ is a semi-norm, that is...
1. it is positive semidefinite: $p_\phi(f) \geq 0$ for all $f \in V$,

2. it is positive homogeneous: $p_\phi(\lambda \cdot f) = \lambda \cdot p_\phi(f)$,

3. and it satisfies the triangle inequality: $p_\phi(f + g) \leq p_\phi(f) + p_\phi(g)$.

Therefore, $V$ is a locally convex topological Hausdorff space.

Now, let $S$ denote the set of all support functions on $E$, which is a subset of $V$. The set $S$ is obviously convex and closed in $V$. Furthermore, $S$ is contained in the product space $\mathbb{R}^E = \prod \{\mathbb{R} : \phi \in E\}$. Define $S[\phi] = \{f(\phi) : f \in S\}$. These sets are bounded for all $\phi \in E$ and their closures $\bar{S}[\phi]$ are therefore compact. By Tychonov’s theorem (Kelley, 1955) the product $\prod \{\bar{S}[\phi] : \phi \in E\}$ is compact and since $S \subseteq \prod \{\bar{S}[\phi] : \phi \in E\}$, $S$ is compact too.

Next we are going to apply the Krein-Milman theorem to the convex, compact set $S$. Here is the theorem:

**Theorem 10.8 Theorem of Krein-Milman:** A convex, compact subset $S$ of a locally convex Hausdorff space is the closed convex hull of its extreme points.

Before we are going to apply this theorem to our problem of finding a random mapping inducing a given support function, we transform the theorem into an integral representation, following (Phelps, 2001). As a preparation we need a new notion. Let $P$ be a probability measure on $S$, that is, a nonnegative regular measure on the $\sigma$-algebra of Borel sets in $S$, such that $P(S) = 1$. A point $f \in V$ is said to be represented by $P$, if for every linear function $h : V \rightarrow \mathbb{R}$,

$$h(f) = \int_S h(v) dP(v).$$

We cite the following lemma from (Phelps, 2001):

**Lemma 10.1** Let $C$ be a compact subset of a locally convex topological space $V$. A point $f \in V$ belongs to the the closed convex hull $H$ of $C$, if and only if there is a probability measure $P$ on $C$ which represents $f$.

Now, with the aid of this lemma we reformulate the Krein-Milman Theorem 10.8.

**Theorem 10.9** Every point $f$ of a convex, compact subset $S$ of a locally convex Hausdorff space $V$ is represented by a probability measure on $S$, which is supported by the closure of the extreme points $\text{ext}(S)$, i.e. $P(\text{ext}(S)) = 1$. 
10.2. GENERATING SUPPORT FUNCTIONS

Proof. By the Krein-Milman Theorem 10.8, \( f \in S \) means that \( f \) belongs to the closure of the convex hull of the extreme points \( \text{ext}(S) \) of \( S \). Clearly, the set of extreme point of \( S \) is bounded, its closure is therefore compact. Hence, by Lemma 10.1, \( f \) is represented by a probability on the closure of the extreme points of \( S \). \( \Box \)

The question is now, what are the extreme point of the set \( S \) of support functions. This question is answered by Theorem 43.4 in (Choquet, 1953–1954). In this theorem Choquet considers functions \( \text{alternating of order } \infty \). This means that in (10.11) of Definition 10.2 the inverse inequality holds. Now, if \( f \) is \( \text{monotone of order } \infty \), then \( g(\phi) = f(0) - f(\phi) \) is \( \text{alternating of order } \infty \). So there is a close relation between the two notions. Choquet further considers alternating functions on an ordered commutative semigroup with a zero-element with all elements greater than zero. This applies to our join-semigroup \( E \), which, in addition, is an \( \text{idempotent} \) semigroup. If \( C \) is a convex cone in \( V \) and \( H \) is an affine subspace of \( V \), not containing the zero function, and which meets every ray of \( C \), then \( C \cap H \) is a convex set and \( f \in C \cap H \) is an extreme point of this convex set, if and only if \( f \) is an extremal point of the convex cone \( C \). As a consequence of Theorem 43.4, Choquet states in Section 46 of (Choquet, 1953–1954) that the extremal points of the convex cone \( M \) of functions monotone to the order \( \infty \) are the exponentials on \( E \), that is functions \( e_{\mathcal{E}} \to \mathbb{R} \) such that \( 0 \leq e(\phi) \leq 1 \), for all \( \phi \in (E) \) and

\[
f(\phi \lor \psi) = f(\phi) \cdot f(\psi).
\]

for all \( \phi, \psi \in \mathcal{E} \).

Note now that item 1 of Definition 10.2 requires for a support function that \( f(0) = 1 \). This defines an affine hyperplane \( H \) in \( V \) and \( M \cap H \) is exactly the set of support functions on \( \mathcal{E} \). So its extreme points are the exponentials on \( \mathcal{E} \) with \( e(0) = 1 \). Since \( \mathcal{E} \) is idempotent, we have for any exponential \( e(\phi) = e(\phi \lor \phi) = e(\phi) \cdot e(\phi) \). Hence \( e(\phi) \) takes only the values 0 or 1. Let \( e_i \) for \( i = 1, 2, \ldots \) be a convergent sequence of exponentials on \( \mathcal{E} \), such that

\[
e(\phi) = \lim_{i \to \infty} e_i(\phi).
\]

Then \( e \) is a support function, since \( S \) is closed, and its is also an exponential on \( \mathcal{E} \). So the set of exponentials is both bounded and closed, hence compact. Define for an exponential \( e \)

\[
I_e = \{ \phi \in \mathcal{E} : e(\phi) = 1 \}.
\]

This is obviously an \( \text{ideal} \) in \( \mathcal{E} \) and any ideal \( I \) in \( \mathcal{E} \) defines an exponential by \( e(\phi) = 1 \) if \( \phi \in I \) and \( e(\phi) = 0 \) otherwise. So, there is a one-to-one relation between exponentials on \( \mathcal{E} \) and ideals of \( \mathcal{E} \). We may identify the set of exponentials by the set \( I_{\mathcal{E}} \) of ideals.
CHAPTER 10. SUPPORT FUNCTIONS

Consider a \( \phi \in \mathcal{E} \). Define, for \( f \in V \), \( h_\phi(f) = f(\phi) \). This defines a continuous linear function \( h_\phi : V \rightarrow \mathbb{R} \). Consider now any support function \( sp \in S \). By the reformulated version of the Krein-Milman Theorem 10.9, \( sp \) is represented by a probability measure on the closed set of its extreme points, that is, the set of exponentials on \( \mathcal{E} \). Hence, we have

\[
sp(\phi) = h_\phi(sp) = \int_{\text{ext}(S)} h_\phi(e) dP(e) = \int_{\text{ext}(S)} e(\phi) dP(e),
\]

for some probability measure \( P \) on \( \text{ext}(S) \) and all \( \phi \in \mathcal{E} \). But, because \( e \) is a 0-1-function, this gives

\[
sp(\phi) = P\{e : e(\phi) = 1\}.
\]

Now, we are nearly done. We consider the probability space \( (\text{ext}(S), B, P) \), where \( B \) denote the Borel \( \sigma \)-algebra of subsets of \( \text{ext}(S) \) and \( P \) the probability introduced above. We now construct a mapping from \( \text{ext}(S) \) into \( (I_\Phi, D) \), the ideal extension of the information algebra \( (\Phi, D) \). Since \( \mathcal{E} \) is supposed to be a sub-semilattice of \( \Phi \), the ideal \( I_e \) associated with the exponential \( e \) can be extended to an ideal in \( \Phi \), generally in may ways, for example by

\[
J_e = \{ \phi \in \Phi : \phi \leq \psi \text{ for some } \psi \in I_e \}.
\]

Then we define the random mapping \( \Gamma(e) = J_e \) from the probability space \( (\text{ext}(S), B, P) \) into the information algebra \( (I_\Phi, D) \). As usual, we consider \( \Phi \) as a subset of \( I_\Phi \) by the embedding \( \phi \mapsto \downarrow \phi \). Consider any \( \phi \in \mathcal{E} \). Then for the support of \( \phi \) by \( \Gamma \) we obtain

\[
\text{s}_\Gamma(\phi) = \{ e \in \text{ext}(S) : \phi \in J_e \} = \{ e \in \text{ext}(S) : \phi \in I_e \} = \{ e \in \text{ext}(S) : e(\phi) = 1 \}.
\]

As we have seen, the last set is measurable, that is belongs to \( B \). Hence we see that all elements of \( \mathcal{E} \) are \( \Gamma \)-measurable, \( \mathcal{E} \subseteq \mathcal{E}_\Gamma \). Further,

\[
sp_\Gamma(\phi) = P(\text{s}_\Gamma(\phi)) = P\{e \in \text{ext}(S) : e(\phi) = 1\} = sp(\phi).
\]

So \( sp_\Gamma \) and \( sp \) coincide on \( \mathcal{E} \). In this sense \( sp \) is induced by the random mapping \( \Gamma \), hence \( \Gamma \) generates \( sp \). We should stress that the \( \Gamma \) defined above is not the unique random mapping generating \( sp \). This issue will be addressed in Section 10.3.

Next we turn to continuous support functions. This time let \( \mathcal{E} \) be a \( \sigma \)-join-semilattice, a semi lattice closed under countable joins. Again, we assume \( \mathcal{E} \) to be a sub-semilattice of some \( \sigma \)-information algebra \( (\Phi, D) \). Let \( S_c \) denote the set of continuous support functions on \( \mathcal{E} \). As above, we argue that \( S_c \) is still a convex, compact subset of the function space \( V \).
Therefore, the revised Theorem of Krein-Milman 10.9 still applies. Because the elements of $S_c$ are still monotone of order $\infty$, Choquet’s Theorem 43.4 (Choquet, 1953–1954) is also still applicable. The extreme elements of $S_c$ are therefore again exponentials on $\mathcal{E}$. But since they belong to $S_c$, they must be continuous exponentials. That is, if $\phi_1 \leq \phi_2 \leq \ldots$ is a monotone sequence in $\mathcal{E}$, then
\[
e(\vee_{i=1}^{\infty} \phi_i) = \lim_{i \to \infty} e(\phi_i).
\]
Since $e$ is a monotone 0-1 function it follows that
\[
e(\vee_{i=1}^{\infty} \phi_i) = \prod_{i=1}^\infty e(\phi_i).
\]
The set of extreme points $ext(S_c)$ is again bounded and closed, hence compact. As above, define $I_e = \{ \phi \in \mathcal{E} : e(\phi) = 1 \}$. This time $I_e$ becomes a $\sigma$-ideal in $\mathcal{E}$.

Consider a continuous support function $sp \in S_c$. Define $h_\phi(f) = f(\phi)$, a linear function from $V$ into $\mathbb{R}$. By Theorem 10.9 there exists a probability measure $P$ on $ext(S_c)$ such that
\[
sp(\phi) = h_\phi(sp) = \int_{ext(S_c)} h_\phi(e) dP(e) = \int_{ext(S_c)} e(\phi) dP(e).
\]
As above this gives
\[
sp(\phi) = P\{ e \in ext(S_c) : e(\phi) = 1 \},
\]
So, again as above, we may define a random mapping from the probability space $(ext(S_c), B_c, P)$ into the ideal completion $(I_\Phi, D)$ of the information algebra $(\Phi, D)$, by $\Gamma(e) = J_e$. Here $B_c$ is the $\sigma$-field of Borel sets in $ext(S_c)$. Note that in this case $J_e$ is a $\sigma$-ideal in $\Phi$. As above we verify that
\[
sp(\phi) = P(\Gamma(e) = 1) = P\{ e \in ext(S_c) : e(\phi) = 1 \} = sp(\phi)
\]
for all $\phi \in \mathcal{E}$. So, $\Gamma$ is a random mapping generating the continuous support function $sp$ on $\mathcal{E}$.

To conclude this part, we formulate the main result of this section in the following theorem

**Theorem 10.10** Let $(\Phi, D)$ be an information algebra and $\mathcal{E} \subseteq \Phi$ a sub-semilattice of $\Phi$ containing $0$. If $sp$ is a support function on $\mathcal{E}$, then there exists a probability space $(\Omega, \mathcal{A}, P)$ and a random mapping $\Gamma$ from this space into the ideal completion of $(I_\Phi, D)$ of $(\Phi, D)$, such that its support function coincides on $\mathcal{E}$, with $sp$, that is $sp(\phi) = sp(\phi)$ for all $\phi \in \mathcal{E}$.

If $(\Phi, D)$ is a $\sigma$-information algebra, $\mathcal{E} \subseteq \Phi$ a $\sigma$-semilattice and $sp$ continuous, then there is a random mapping $\Gamma$ generating $sp$, which maps to $\sigma$-ideals of $\Phi$. 

For continuous support functions there is an alternative approach to generate it from a random mapping, due to (Norberg, 1989). We consider now continuous support functions on a continuous information algebras (not necessarily $D$-continuous, see Section 6.4). The important point is that in this case $\Phi$ is a continuous lattice (Theorem 6.17). We consider the Scott topology in $\Phi$ (see Section 8.3). This topology is called second countable, if it has a countable topological basis. Then, let $\Sigma(\Phi)$ be the minimal $\sigma$-field over the sets $\uparrow \phi = \{ \psi \in \Phi : \phi \leq \psi \}$. According to (Norberg, 1989) $\Sigma(\Phi)$ is the Borel-$\sigma$-field with respect to the Scott topology. Now, Theorem 3.3 of (Norberg, 1989) can be stated in our context as follows:

**Theorem 10.11** If $\Phi$ is a continuous lattice with a least element $0$ and the Scott topology of $\Phi$ is second countable, then

$$sp(\phi) = P(\uparrow \phi)$$

defines a bijection between continuous support functions on $\Phi$ and probability measures $P$ on $\Sigma(\Phi)$.

Hence, if we consider any continuous support function $sp$ on $\Phi$, then the identity mapping $id$ on $\Phi$ defines a random mapping form the probability space $(\Phi, \Sigma(\Phi), P)$ into $\Phi$. Obviously, we have for the support of $\phi$ under this random mapping $sp_{id}(\phi) = \uparrow \phi$, hence

$$sp_{id}(\phi) = P(\uparrow \phi) = sp(\phi).$$

Thus, the random mapping $id$ on the probability space $(\Phi, \Sigma(\Phi), P)$ indeed generates the support function $sp$. Note that, according to Theorem 10.11, the probability measure $P$ is determined uniquely by $sp$.

As an application assume the $(\Phi, D)$ is a compact information algebra with a countable set of $\Phi_f$ of finite elements. The sets $\uparrow \psi$ for $\psi \in \Phi_f$ form a basis for the Scott topology in $\Phi$ (Section 8.3), which is therefore second countable. Thus, Theorem 10.11 applies. Now, clearly

$$\uparrow \phi = \bigcap_{\psi \in \Phi_f, \psi \leq \phi} \uparrow \psi.$$

Since $\Phi_f$ is countable, so is the set $\{ \psi \in \Phi_f, \psi \leq \phi \}$ and we may number its elements by $\psi_1, \psi_2, \ldots$. If we define $\eta_i = \bigvee_{j=1}^{i} \psi_j$, we obtain a monotone sequence of finite elements and

$$\uparrow \phi = \bigcap_{i=1}^{\infty} \uparrow \eta_i.$$

Due to the monotonicity of the sequence $\eta_i$, we have $\uparrow \eta_1 \supseteq \uparrow \eta_2 \supseteq \ldots$. Therefore the continuity of probability implies that

$$sp(\phi) = P(\uparrow \phi) = \lim_{i \to \infty} P(\uparrow \eta_i) = \lim_{i \to \infty} sp(\eta_i). \quad (10.14)$$
This shows that a continuous support function on a compact information algebra with countably many finite elements is fully determined by its values for finite elements. A similar result holds also for a continuous information algebra with a countable basis: A continuous support function is determined by its values for the basis elements.

### 10.3  Canonical Random mappings

According to the previous Section 10.2 any support function can be generated by some random mapping. In this section we are going to examine the random mappings generating a given support function in more detail. In particular, we shall compare these random mappings and single out a particular one, which we shall call the **canonical** mapping.

Let \((\Phi, D)\) be an information algebra and \(E \subseteq \Phi\) a sub-semilattice of \(\Phi\), containing \(0\). Consider a support function \(\text{sp}\) on \(E\). According to the discussion in Section 10.2 there is a probability space \((\text{ext}(S), \mathcal{A}, P)\) on the set of exponentials \(\text{ext}(S)\) on \(E\) and a random mapping into the **ideal completion** of \((\Phi, D)\) defined by

\[
\nu(e) = \{ \phi \in \Phi : \phi \leq \psi \text{ for some } \psi \in I_e \}
\]

where \(I_e\) is the ideal \(\{ \psi \in E : e(\psi) = 1 \}\) in \(E\) associated with the exponential \(e\). Then we obtain for \(\psi \in E\)

\[
\text{sp} (\psi) = P\{ I \in I_E : \psi \in I \},
\]

which shows that the random mapping \(\nu\) from \(\text{ext}(S)\) into the ideal completion \(I_\Phi\) of \(\Phi\) indeed generates the support function on \(E\).

We noted in Section 10.2 that this random mapping \(\nu\) is not the only one inducing the support function on \(E\). Let’s examine this more in detail. The restriction of an ideal \(I\) of \(\Phi\) to \(E\) is clearly an ideal of \(E\). We define the mapping \(p : I_\Phi \to I_E\) by \(p(I) = I|E\); to each ideal in \(\Phi\), we associate its restriction to \(E\). The inverse mapping \(p^{-1}(I) = \{ J \in I_\Phi : p(J) = I \}\) induces a partition of \(I_\Phi\). Consider any ideal \(J \in p^{-1}(I)\). We claim that
Let’s pursue this observation. Consider the probability algebra \((B, \mu)\) associated with the probability space \((I_\mathcal{E}, \mathcal{A}, P)\) (see Section 9.4). We remind that the mapping \(\rho_\nu = \rho_0 \circ s_\nu\) from \(\Phi\) into \(B\) is an allocation of probability (a.o.p.) (see Section 9.4). This a.o.p. as every a.o.p. on \(\Phi\) induces a support function \(sp_\nu = \mu \circ \rho_0 \circ s_\nu\) is a support function on \(\Phi\) (see Theorem 10.5), and its restriction to \(\mathcal{E}\) equals \(sp\). So, \(sp_\nu\) is an extension of \(sp\) to \(\Phi\). Now, for any random mapping \(\Gamma\) from \(I_\mathcal{E}\) into the ideal completion \(I_\Phi\) of \(\Phi\), such that \(\Gamma(I) \in p^{-1}(I)\), we have \(\nu(I) \subseteq \Gamma(I)\). This implies for the allocations of support that \(s_\nu(\phi) \subseteq s_\Gamma(\phi)\), hence \(\rho_\nu(\phi) = \rho_0(s_\nu(\phi)) \leq \rho_0(s_\Gamma(\phi)) = \rho_\Gamma(\phi)\) and for \(\psi \in \mathcal{E}\), we have \(\rho_\nu(\psi) = \rho_\Gamma(\psi)\). It follows that

\[
sp_\nu(\phi) = \mu(\rho_0(s_\nu(\phi))) \leq \mu(\rho_0(s_\Gamma(\phi))) = sp_\Gamma(\phi)
\]

We shall see later (Section 10.4) that the random mapping \(\rho_\nu\) generates the least extension of the support function \(sp\) on \(\mathcal{E}\) to \(\Phi\) among all extensions. But before we turn to this question, we return to the random mappings generating \(sp\) on \(\mathcal{E}\).

Consider the family of sets \(\{I \in I_\mathcal{E} : \psi \in I\}\) for \(\psi \in \mathcal{E}\). All these sets belong to the \(\sigma\)-field \(\mathcal{A}\) in the probability space \((I_\mathcal{E}, \mathcal{A}, P)\) used to define the random mapping \(\nu\) to generate the support function \(sp\) on \(\mathcal{E}\). This is so, because \(sp(\psi)\) equals the probability \(P\) of these sets, and they must thus be measurable. Let \(\mathcal{A}_\mathcal{E} \subseteq \mathcal{A}\) be the \(\sigma\)-fields of subsets generated by the family of these subsets. Note that this set depends only on the semi lattice \(\mathcal{E}\), but not on \(sp\) itself. Denote the restriction of the probability measure \(P\) to \(\mathcal{A}_\mathcal{E}\) by \(P_{sp}\). This probability depends on the support function \(sp\), and thereby indirectly of course also on \(\mathcal{E}\). Consider the probability space \((I_\mathcal{E}, \mathcal{A}_\mathcal{E}, P_{sp})\). We remark that the random mapping \(\nu\), as well as the related mappings \(\Gamma\) considered above, still generate \(sp\) on \(\mathcal{E}\).

In order to improve comparisons between random mappings generating the support function \(sp\) on \(\mathcal{E}\), we transport probability from the set of ideal \(I_\mathcal{E}\) in \(\mathcal{E}\) to the set \(I_\Phi\) of ideals in \(\Phi\). The family of sets \(p^{-1}(A)\) for \(A \in \mathcal{A}_\mathcal{E}\)
forms a $\sigma$-field of subsets of $I_\Phi$ and by $P(p^{-1}(A)) = P_{sp}(A)$ a probability measure is defined on this $\sigma$-field. By abuse of notation, we denote the new probability space by $(I_\Phi, A_\mathcal{E}, P_{sp})$. The random mapping $\nu$ from $I_\mathcal{E}$ into the ideal completion of $\Phi$ is redefined as $\nu(p(I))$ for $I \in I_\Phi$. Again, we call this new mapping $\nu$, that is,

$$\nu(I) = \{ \phi \in \Phi : \phi \leq \psi \text{ for some } \psi \in p(I) \}. \quad (10.15)$$

We call this random mapping $\nu$, together with the associated probability space $(I_\Phi, A_\mathcal{E}, P_{sp})$, the canonical random mapping generating the support function $sp$ on the semilattice $\mathcal{E}$. Any other random mapping $\Gamma$ defined above of $I_\mathcal{E}$ may similarly be redefined as $\Gamma(p(I))$.

We may now compare different extensions of support functions from $\mathcal{E}$. Consider semilattices $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \Phi$ and support functions $sp_1$ and $sp_2$ on $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively, such that $sp_2$ is an extension of $sp_1$. Then, both of these support functions have their canonical random mappings $\nu_1$ and $\nu_2$ defined on the probability spaces $(I_\Phi, A_{\mathcal{E}_1}, P_{sp_1})$ and $(I_\Phi, A_{\mathcal{E}_2}, P_{sp_2})$ respectively. The next theorem shows how these canonical random mappings are related.

**Theorem 10.12** Let $(\Phi, D)$ be an information algebra and let $\nu_1$ and $\nu_2$, defined on the probability spaces $(I_\Phi, A_{\mathcal{E}_1}, P_{sp_1})$ and $(I_\Phi, A_{\mathcal{E}_2}, P_{sp_2})$, be the canonical random mappings associated with the support functions $sp_1$ and $sp_2$ on the semilattices $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \Phi$. If $sp_2$ is an extension of $sp_1$, that is $sp_1 = sp_2|\mathcal{E}_2$, then

1. $\nu_1 \leq \nu_2$, in the order of the information algebra of random mappings into $(\Phi, D)$,
2. $A_{\mathcal{E}_1} \subseteq A_{\mathcal{E}_2}$,
3. $P_{sp_1} = P_{sp_2}|A_{\mathcal{E}_1}$, on $A_{\mathcal{E}_1}$ the two probability measures are equal.
4. $sp_{\nu_1} \leq sp_{\nu_2}$.

**Proof.** (1) By definition we have $p_1(I) = I|\mathcal{E}_1$ and $p_2(I) = I|\mathcal{E}_2$, hence $p_1(I) \subseteq p_2(I)$. Therefore, from (10.15), we conclude that $\nu_1(I) \subseteq \nu_2(I)$ for all $I \in I_\Phi$, hence $\nu_1 \leq \nu_2$.

(2) Consider an element $\psi \in \mathcal{E}_1$. Then, the allocations of support relative to $\nu_1$ and $\nu_2$, respectively, are

$$s_{\nu_1}(\psi) = \{ I \in I_\Phi : \psi \in \nu_1(I) \} = \{ I \in I_\Phi : \psi \in I|\mathcal{E}_1 \},$$

$$s_{\nu_2}(\psi) = \{ I \in I_\Phi : \psi \in \nu_2(I) \} = \{ I \in I_\Phi : \psi \in I|\mathcal{E}_2 \}.$$ 

But $\psi \in I|\mathcal{E}_1$ implies $\psi \in I|\mathcal{E}_2$. On the other hand, $\psi \in \mathcal{E}_1$ and $\psi \in I|\mathcal{E}_2$ implies $\psi \in I|\mathcal{E}_2 \cap \mathcal{E}_1 = I|\mathcal{E}_1$. So, we conclude that $s_{\nu_1}(\psi) = s_{\nu_2}(\psi)$ for every
Dynkin calls a family of sets, closed under finite intersections, a \( s \)-system. The family \( P \) of sets \( s_{v_1}(\psi) \) for \( \psi \in \mathcal{E}_1 \) is a \( \pi \)-system (see Theorem 10.1). The family \( L \) of sets \( A \in \mathcal{A}_E \) for which

\[
P_{sp_1}(A) = P_{sp_2}(A)
\]

is closed under complementation, and contains \( \cup_i A_i \) if \( A_i \) is a family of disjoint sets in \( L \). This is called a \( \lambda \)-system by Dynkin. From the consideration above, we see that \( P \subseteq L \). The theorem of Dynkin states that if \( P \) is a \( \pi \)-system and \( L \) a \( \lambda \)-system, then \( P \subseteq L \) implies that the \( \sigma \)-closure of \( P \) is contained in \( L \), that is \( \sigma(P) \subseteq L \). In our case the \( \sigma \)-closure of \( P \) is \( \mathcal{A}_E \), hence we have \( \mathcal{A}_E \subseteq L \), where \( L \) contains all sets of \( \mathcal{A}_E \) on which the two probabilities coincide. So, indeed for all \( A \in \mathcal{A}_E \) we have \( P_{sp_1}(A) = P_{sp_2}(A) \).

(4) We have (see (9.15)) \( sp_{v_1}(\phi) = sp_{v_2}(s_{v_1}(\phi)) \leq P_{sp_2}(s_{v_2}(\phi)) = sp_{v_2}(\phi) \), because \( s_{v_1} \subseteq s_{v_2} \) and also \( P_{sp_1}(A) \leq P_{sp_2}(A) \) for any set \( A \in I_{\Phi} \).

Therefore, \( sp_{v_1} \leq sp_{v_2} \). \( \square \)

This theorem shows in particular, that the canonical random mapping associated with a support function \( sp \) on a semilattice \( \mathcal{E} \subseteq \Phi \) is unique. It permits also to conclude that \( sp_{\nu} \) is indeed the least extension of the support function \( sp \) from \( \mathcal{E} \) to \( \Phi \). Indeed, suppose that \( sp' \) is any extension of \( sp \) to \( \Phi \). Then, \( sp' \) is generated by a canonical random mapping \( \nu' \). According to Theorem 10.12 (4) we have then

\[
sp_{\nu} \leq sp_{\nu'} = sp'.
\]

The last equity holds because \( sp' \) is defined on \( \Phi \). So, we have

**Corollary 10.1** If \( sp \) is a support function defined on a semilattice \( \mathcal{E} \subseteq \Phi \), then \( sp_{\nu} \) is the least extension of \( sp \) to \( \Phi \), that is, \( sp_{\nu} \leq sp' \) for any support function \( sp' \) on \( \Phi \) such that \( sp = sp'|E \).

We remark, that a similar analysis can be done for \( \sigma \) semilattices or complete lattices \( \mathcal{E} \) and continuous or condensable support functions \( sp \). However, more interesting is the case of compact information algebras \((\Phi, D)\). As above, we assume that the finite elements \((\Phi_f, D)\) form a subalgebra of \((\Phi, D)\). We consider a support \( sp \) defined on \( \Phi_f \), thus \( \mathcal{E} = \Phi_f \). Since in this case \((\Phi_f, D)\) its ideal completion \((I_{\Phi_f}, D)\) is isomorphic to \((\Phi, D)\) we identify ideals \( I \) of \( \Phi_f \) with its supremum \( \vee I \in \Phi \). For the support function \( sp \), we consider its canonical probability space \((I_{\Phi_f}, \mathcal{A}_{\Phi_f}, P_{sp})\).

Instead of the canonical random mapping,

\[
\nu(I) = \{ \phi \in I : \phi \leq \psi \text{ for some } \psi \in I \}
\]
we consider also the random mappings
\[ 
\sigma(I) = \{ \phi \in I : \phi \leq \bigvee_{i=1}^{\infty} \psi_i, \psi_i \in I \}, \quad (10.16) 
\]
\[ 
\gamma(I) = \downarrow \bigvee I. \quad (10.17) 
\]
Both map \( I_{\Phi} \) into \( I_{\Phi} \). We are going to examine the support functions on \( \Phi \) induced by these random mappings.

We start with the random mapping \( \sigma \). Here are its basic properties:

**Lemma 10.2** Let \( (\Phi, D) \) be a compact information algebra, its finite elements \((\Phi_f, D)\) a subalgebra and \( \sigma \) the random map defined by (10.16)

1. The ideal \( \sigma(I) \) is a \( \sigma \)-ideal in \( \Phi \).
2. Its restriction to \( \Phi_f \) equals \( I \), \( \sigma(I)|_{\Phi_f} = I \).
3. The \( \sigma \)-ideal \( \sigma(I) \) is minimal among all \( \sigma \)-ideals in \( \Phi \) extending \( I \).

**Proof.** (1) Assume \( \phi_1, \phi_2, \ldots \in \sigma(I) \). Then we have \( \phi_i \leq \bigvee_{j=1}^{\infty} \psi_{i,j} \) with \( \psi_{i,j} \in I \) for all \( i = 1, 2, \ldots \) and \( j = 1, 2, \ldots \). But then we obtain
\[
\bigvee_{i=1}^{\infty} \phi_i = \bigvee_{i=1}^{\infty} \psi_{i,j} = \bigvee_{i=1}^{\infty} \psi_{i,j}',
\]
where \( \psi_{i,j}' = \bigvee_{j=1}^{h} \psi_{i,j} \in I \). This shows that \( \bigvee_{i=1}^{\infty} \in \sigma(I) \), hence \( \sigma(i) \) is indeed a \( \sigma \)-ideal in \( \Phi \).

(2) Assume that \( \psi \in \sigma(I) \) and \( \psi \in \Phi_f \). Then \( \psi \leq \bigvee_{i=1}^{\infty} \psi_i \), with \( \psi_i \in I \) for \( i = 1, 2, \ldots \). By the usual transformation, we may always assume that \( \psi_1 \leq \psi_2 \leq \ldots \). This monotone sequence is a directed set in \( \Phi \). By compactness there exists a \( \psi_i \) such that \( \psi \leq \psi_i \). This shows that \( \psi \in I \). But clearly, \( I \subseteq \sigma(I) \), therefore we see that indeed the restriction of \( \sigma(I) \) to \( \Phi_f \) equals \( I \).

(3) Consider a \( \sigma \)-ideal whose restriction to \( \Phi_f \) equals \( I \). Assume \( \phi \in \sigma(I) \). The \( \phi \leq \bigvee_{i=1}^{\infty} \psi_i \), with the \( \psi_i \) in \( I \), hence in \( J \). But then \( \bigvee_{i=1}^{\infty} \psi_i \in J \) since \( J \) is a \( \sigma \)-ideal, therefore \( \phi \in J \). This shows that \( \sigma(I) \subseteq J \). Hence \( \sigma(I) \) is indeed minimal among the \( \sigma \)-ideals extending \( I \). \( \square \)

The random map \( \sigma \) generates a support function \( sp_\sigma = \mu \circ \rho_\sigma \) on \( \Phi \), where as usual \((\mu, B)\) is the probability algebra associated with the probability space \((I_{\Phi_f}, A_{\Phi_f}, P_{sp})\), and \( \rho_\sigma = \rho_0 \circ s_\sigma \). We are going to show that \( sp_\sigma \) is a continuous extension of \( sp \). The key is in the following lemma:

**Lemma 10.3** Let \( (\Phi, D) \) be a compact information algebra, its finite elements \((\Phi_f, D)\) a subalgebra and \( \sigma \) a random map defined by (10.16) and \( s_\sigma \) the allocation of support for the random map \( \sigma \). Then, if \( \phi_i \in \Phi \) for \( i = 1, 2, \ldots \),
\[
s_\sigma\left(\bigvee_{i=1}^{\infty} \phi_i\right) = \bigcap_{i=1}^{\infty} s_\sigma(\phi_i).
\]
CHAPTER 10. SUPPORT FUNCTIONS

Proof. Since $\Phi$ is a complete lattice, $\bigvee_{i=1}^{\infty} \phi_i \in \Phi$, and

$$s_\sigma(\bigvee_{i=1}^{\infty} \phi_i) = \{ I \in I_{\Phi_f} : \bigvee_{i=1}^{\infty} \phi_i \leq \bigvee_{i=1}^{\infty} \psi_i, \psi_i \in I \}.$$  

If $I \in s_\sigma(\bigvee_{i=1}^{\infty} \phi_i)$. Then clearly $I \in s_\sigma(\phi_i)$ for all $i = 1, 2, \ldots$ Conversely, assume $I \in s_\sigma(\phi_i)$ for all $i = 1, 2, \ldots$. Then we have $\phi_i \leq \bigvee_{j=1}^{\infty} \psi_{i,j}$ with $\psi_{i,j} \in I$. This implies in the same way as in the proof of Lemma 10.2 that $\bigvee_{i=1}^{\infty} \phi_i \in \sigma(I)$, hence $I \in s_\sigma(\bigvee_{i=1}^{\infty} \phi_i)$ and this proves the lemma. \(\square\)

As a consequence of this lemma, we find that

$$\rho_\sigma(\bigvee_{i=1}^{\infty} \phi_i) = \rho_0(\bigvee_{i=1}^{\infty} \phi_i) = \rho_0(\bigvee_{i=1}^{\infty} s_\sigma(\phi_i))$$

$$= \bigwedge_{i=1}^{\infty} \rho_0(s_\sigma(\phi_i)) = \bigwedge_{i=1}^{\infty} \rho_\sigma(\phi_i).$$

(10.18)

The allocation of probability $\rho_\sigma$ is a $\sigma$-a.o.p. By Theorem 10.5 $sp_\sigma$ is a continuous support function extending $sp$ on $\Phi_f$ to $\Phi$.

Let’s fix this result in the following theorem:

**Theorem 10.13** Let $(\Phi, D)$ be a compact information algebra, its finite elements $(\Phi_f, D)$ a subalgebra, $sp$ a support function defined on $\Phi_f$ and $\sigma$ a random map defined by (10.16). Then, if $(\mu, B)$ is the probability algebra associated with the canonical probability space $(I_{\Phi_f}, A_{\Phi_f}, P_{sp})$ and $\rho_\sigma = \rho_0 \circ s_\sigma$, then $sp_\sigma = \mu \circ \rho_\sigma$ is the continuous extension of $sp$ to $\Phi$.

We turn to the random mapping $\gamma$, define in (10.17). This mapping is characterized as follows:

**Lemma 10.4** Let $(\Phi, D)$ be a compact information algebra, its finite elements $(\Phi_f, D)$ a subalgebra and $\gamma$ the random mapping defined by (10.17). Then the ideal $\gamma(I)$ is the minimal complete ideal in $\Phi$ whose restriction to $\Phi_f$ equals $I$, $\gamma(i)|\Phi_f = I$.

Proof. Clearly $I \subseteq \downarrow \bigvee I \cap \Phi_f$. Consider then an element $\psi \in \downarrow \bigvee I \cap \Phi_f$. From $\psi \leq \bigvee I$ it follows, since $I$ is a directed set, by compactness that there is a $\chi \in I$ such that $\psi \leq \chi$. But then $\psi \in \Phi_f$ implies $\psi \in I$. This proves that $\gamma(i)|\Phi_f = I$.

As a principal ideal in a complete lattice $\gamma(I)$ is a complete ideal. Consider any other complete ideal $J$, that is $J = \downarrow \bigvee J$ whose restriction to $\Phi_f$ equals $I$. But then $\bigvee I \leq \bigvee J$, hence $\gamma(I) \subseteq J$. This proves the minimality of $\gamma(I)$. \(\square\)

Consider now simple random variables $\Delta$ on the canonical probability space $(I_{\Phi_f}, A_{\Phi_f}, P_{sp})$. Any such random variable is defined by a partition $B_i, i = 1, \ldots, m$ of $I_{\Phi_f}$, and $\Delta(I) = \phi_i \in \Phi$ if $I \in B_i$. Note that $\Delta \leq \gamma$ if and only if $\phi_i \leq \bigvee I$ for $I \in B_i$ and $i = 1, \ldots, m$. This leads to the following result:
10.3. CANONICAL RANDOM MAPPINGS

Lemma 10.5 The random mapping $\gamma$ defined by (10.17) is a generalized random variable,

$$\gamma = \vee\{\Delta : \Delta \text{ simple random variable}, \Delta \leq \gamma\}.$$  

Proof. We claim that for all $I \in I_{\Phi_f}$ we have $\gamma(I) = \vee\{\Delta(I) : \Delta \leq \gamma\}$ where it is understood that $\Delta$ denote simple random variables. Clearly $\gamma(I) \geq \vee\{\Delta(I) : \Delta \leq \gamma\}$. To prove the converse inequality, consider $I \in I_{\Phi_f}$. Then we have by density $\gamma(I) = \vee\{\psi \in \Phi_f : \psi \leq \vee I\}$. But $\psi \leq \vee I$ implies that there is a $\phi \in I$ such that $\psi \leq \phi$. But then $\psi \in /\Phi_f$ implies that $\psi \in I$. So, $\psi \in I$ if and only if $\psi \leq \vee I$ and $\psi \in \Phi_f$. Define, for a $\psi \in I$,

$$\Delta_{\psi}(I) = \begin{cases} 
\psi & \text{if } \psi \in I, \\
0 & \text{otherwise}.
\end{cases}$$

The set $\{I : \psi \in I\}$ is measurable (belongs to $A_{\Phi_f}$), hence $\Delta_{\psi}$ is a simple random variable and $\Delta_{\psi}(I) \leq \vee I$, hence $\Delta_{\psi} \leq \gamma$. Thus, we conclude

$$\gamma(I) = \vee\{\Delta_{\psi}(I) : \psi \in \Phi_f, \psi \in I\} \leq \vee\{\Delta(I) : \Delta \leq \gamma\}.$$  

This proves the identity $\gamma(I) = \vee\{\Delta(I) : \Delta \leq \gamma\}$, hence the lemma. $\square$

From this lemma it follows that according to Theorem 9.15 we have that

$$\rho_\gamma(\vee X) = \bigwedge_{\phi \in X} \rho_\gamma(\phi).$$

Further, by Theorem 10.5, then

$$sp_\gamma(\vee X) = \inf_{\phi \in X} sp_\gamma(\phi).$$

This means also that

$$sp_\gamma(\phi) = \inf_{\phi \in X} \{sp_\gamma(\psi) : \psi \in \Phi_f, \psi \leq \phi\}. \tag{10.19}$$

That means that $sp_\gamma$ is the unique condensable extension of $sp$ to $\Phi$. We note also that according to Theorem 10.12, since $\nu \leq \sigma \leq \gamma$, we have $sp_\nu(\phi) \leq sp_\sigma(\phi) \leq sp_\gamma(\phi)$. Since $\sigma(I)$ is the least $\sigma$-ideal among all $\sigma$-ideals extending the ideal $I$ of $\Phi_f$ to $\Phi$, we conclude that $sp_\sigma$ is also the minimal continuous support function among all continuous support functions $sp$ extending $sp$ from $\Phi_f$ to $\Phi$,

$$sp_\sigma \leq \tilde{sp}(\phi), \text{ if } \tilde{sp} \text{ continuous, } \tilde{sp}|_{\Phi_f} = sp$$

for all $\phi \in \Phi$. These results (Theorem 10.13 and (10.19)) partly answer an open question posed in (Shafer, 1979). In this work it was shown that continuous and condensable extensions always exist if $E$ is a subset lattice. Here it is shown that they always exist if $E$ corresponds to the finite elements
of a compact information algebra, independently whether \( \Phi \) is a lattice or not.

We conclude by proving the converse of Theorem 10.5 and thus characterizing continuous and condensable support functions by their associated allocations of support.

**Theorem 10.14** 1. If \( \Phi \) is a \( \sigma \)-semilattice, then \( sp = \mu \circ \rho \) is continuous on \( \Phi \) if and only if \( \rho \) is a \( \sigma \)-allocation of probability, that is for \( \phi_i \in \Phi \), \( i = 1, 2, \ldots \)

\[
\rho(\lor_{i=1}^\infty \phi_i) = \land_{i=1}^\infty \rho(\phi_i). \tag{10.20}
\]

2. If \( \Phi \) is a complete lattice, then \( sp = \mu \circ \rho \) is condensable on \( \Phi \) if and only if for any directed set \( X \subseteq \Phi \),

\[
\rho(\lor X) = \land_{\phi \in X} \rho(\phi). \tag{10.21}
\]

**Proof.** The if-part of both parts is already proved in Theorem 10.5, it remains thus only to prove the only-if-part

(1) Consider a countable set of elements \( \phi_1, \phi_2, \ldots \in \Phi \). We may always replace this sequence by a monotonically increasing sequence \( \phi_1' \leq \phi_2' \leq \ldots \) having the same supremum, \( \lor_{i=1}^\infty \phi_i = \lor_{i=1}^\infty \phi_i' \), by defining \( \phi_i' = \lor_{j=1}^i \phi_j \). Then \( \rho(\phi_i') \leq \rho(\phi_i) \leq \ldots \) is downwards directed. Therefore, by the continuity of \( sp \) and Lemma 9.1,

\[
sp(\lor_{i=1}^\infty \phi_i) = sp(\lor_{i=1}^\infty \phi_i') = \lim_{i \to \infty} sp(\phi_i') = \lim_{i \to \infty} \mu(\land_{i=1}^\infty \rho(\phi_i')) = \mu(\land_{i=1}^\infty \rho(\phi_i)).
\]

From \( sp(\lor_{i=1}^\infty \phi_i) = \mu(\land_{i=1}^\infty \rho(\phi_i)) \) it follows that \( \mu(\land_{i=1}^\infty \rho(\phi_i)) = \mu(\lor_{i=1}^\infty \rho(\phi_i)) \). Since clearly \( \land_{i=1}^\infty \rho(\phi_i) \geq \rho(\lor_{i=1}^\infty \phi_i) \) and \( \mu \) is a positive measure, it follows indeed that \( \land_{i=1}^\infty \rho(\phi_i) = \rho(\lor_{i=1}^\infty \phi_i) \).

(2) Let \( X \subseteq \Phi \) be directed. By the condensability of \( sp \) we obtain

\[
\mu(\rho(\lor X)) = sp(\lor X) = \inf_{\phi \in X} sp(\phi) = \inf_{\phi \in X} \mu(\rho(\phi)).
\]

Since the set \( \{ \rho(\phi) : \phi \in X \} \) is downwards directed, we get by Lemma 9.1 \( \inf_{\phi \in X} \mu(\rho(\phi)) = \mu(\land_{\phi \in X} \rho(\phi)) \), hence \( \mu(\rho(\lor X)) = \mu(\land_{\phi \in X} \rho(\phi)) \). Since \( \land_{\phi \in X} \rho(\phi) \geq \rho(\lor X) \), we conclude that \( \land_{\phi \in X} \rho(\phi) = \rho(\lor X) \) \( \square \)

If \( \Phi, D \) is a compact information algebra and \( sp \) condensable on \( \Phi \), then (10.21) implies also that for all \( \phi \in \Phi \)

\[
\rho(\phi) = \land \{ \rho(\psi) : \psi \in \Phi_f, \psi \leq \phi \}.
\]

We are going to study these different extensions of a support functions from a part of \( \Phi \) to the whole of \( \Phi \) in the next section from a different angle.
### 10.4 Minimal Extension

In the previous section, we have found an extension \( sp_f \) for any support function \( sp \) on some sub-semilattice \( E \) of an information algebra \( (\Phi, D) \). This extension is defined in terms of the canonical random mapping of \( sp \). In this section, we shall show how the extension \( sp_f \) can be defined explicitly in terms of the support function \( sp \) on \( E \). This result is due to (Shafer, 1973).

**Theorem 10.15** If \( sp \) is a support function defined on a semilattice \( E \subseteq \Phi \), then

\[
sp_f(\phi) = \sup \left\{ \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} sp(\bigvee_{i \in I} \phi_i) \right\}
\]

(10.22)

where the supremum is to be taken over all elements \( \phi_1, \ldots, \phi_n \geq \phi, n = 1, 2, \ldots \) with \( \phi_1, \ldots, \phi_n \in E \).

**Proof.** Let \( f \) denote the function on the right hand side of (10.22). We remark that \( f \) is equal to \( sp \) on \( E \). Note also that \( f \) is less or at most equal to \( sp_f \), since the latter, as a support function on \( \Phi \), is monotone of order \( \infty \). Therefore, it is sufficient to show that \( f \) is a support function on \( \Phi \), because then, according to Corollary 10.1 it must be greater or equal to \( sp_f \), so that \( sp_f = f \) as claimed.

In order to prove \( f \) to be a support function, we use, following (Shafer, 1973), with allocations of support. Let \( \rho \) be the allocation of support associated with the canonical random mapping generating \( sp \), such that for \( \phi \in E \),

\[
sp(\phi) = \mu(\rho(\phi)),
\]

where \( \mu \) is the probability of the probability algebra \( (B, \mu) \) associated with the probability space \( (I_\Phi, A_\Phi, P_sp) \) of the canonical probability space of the support function \( sp \) on \( E \). Further, \( \rho = \rho_0 \circ sp \) (compare Section 9.4). Define

\[
\bar{\rho}(\phi) = \bigvee \{ \rho(\psi) : \psi \in E, \phi \leq \psi \}.
\]

(10.23)

We are going to show that \( \bar{\rho} \) is an a.o.p. Obviously, for \( \psi \in E \), we have \( \bar{\rho}(\psi) = \rho(\psi) \), hence in particular \( \bar{\rho}(\emptyset) = \top \). Consider \( \phi_1, \phi_2 \in \Phi \). Then \( \phi_1, \phi_2 \leq \phi_1 \lor \phi_2 \), hence \( \bar{\rho}(\phi_1, \phi_2) \geq \bar{\rho}(\phi_1 \lor \phi_2) \) or \( \bar{\rho}(\phi_1) \lor \bar{\rho}(\phi_2) \geq \bar{\rho}(\phi_1 \lor \phi_2) \).

On the other hand, let \( \psi_1 \leq \psi_1 \lor \psi_2 \leq \psi_2 \leq \psi_2 \lor \psi_2 \leq \psi_2 \leq \psi_2 \lor \psi_2 \). Then, \( \psi_1 \lor \psi_2 \leq E \) and \( \phi_1 \lor \phi_2 \leq \psi_1 \lor \psi_2 \lor \psi_2 \) such that \( \rho(\psi_1) \land \rho(\psi_2) = \rho(\psi_1 \lor \psi_2) \leq \bar{\rho}(\phi_1 \lor \phi_2) \).

It follows that

\[
\bar{\rho}(\phi_1 \lor \phi_2) \geq \bigvee \{ \rho(\psi_1) \land \rho(\psi_2) : \phi_1 \leq \psi_1, \phi_2 \leq \psi_2, \psi_1, \psi_2 \in E \} = \bigvee \{ \rho(\psi_1) : \phi_1 \leq \psi_1 \lor \psi_2 \} \land \bigvee \{ \rho(\psi_2) : \phi_2 \leq \psi_1 \lor \psi_2 \} = \bar{\rho}(\phi_1) \land \bar{\rho}(\phi_2).
\]
So, we conclude that $\bar{\rho}(\phi_1 \lor \phi_2) = \bar{\rho} \phi_1 \land \bar{\rho}(\phi_2)$ and that, therefore, $\bar{\rho}$ is an a.o.p.

In the formula (10.22) for $f$, we may replace $sp$ by $\mu \circ \rho$,

$$f = \sup \left\{ \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} \mu(\rho(\bigvee_{i \in I} \phi_i)) \right\}$$

$$= \sup \left\{ \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} \mu(\bigwedge_{i \in I} \rho(\phi_i)) \right\}$$

$$= \sup \{ \mu(\bigvee_{i=1}^n \rho(\phi_i)) \}$$

(10.24)

by the inclusion-exclusion-formula of probability theory. The supremum carries over the same range as in (10.22). The family of elements $\bigvee_{i=1}^n \rho(\phi_i)$ in this supremum forms an upwards directed set in $B$. By Lemma 9.1 we obtain therefore

$$f(\phi) = \mu(\bigvee \{ \bigvee_{i=1}^n \rho(\phi_i) : \phi_i \in \mathcal{E}, \phi_i \geq \phi, i = 1, \ldots, n, n = 1, 2, \ldots \})$$

$$= \mu(\{ \rho(\psi) : \psi \in \mathcal{E}, \psi \geq \phi \})$$

$$= \mu(\bar{\rho}(\phi))$$

Here, the associate law for joins in a complete lattice is used. Since $\bar{\rho}$ is an a.o.p., $f = \mu \circ \bar{\rho}$ is a support function on $\Phi$ (see Theorem 10.5). This concludes the proof. $\square$

In the proof above we used the a.o.p. $\rho$ associated with the support function $sp$ on $\mathcal{E}$. We remind that $sp_\nu = \mu \circ \rho = \mu \circ s_0 \circ s_\nu$. On the other hand the a.o.p. $\bar{\rho}$ generates $f$, that is $f = \mu \circ \bar{\rho}$. From $sp_\nu = f$, as stated in the theorem, we deduce as a complement that $\rho = \rho_0 \circ s_\nu = \bar{\rho}$. In fact, we have seen that for $\psi \in \mathcal{E}$ we have $\rho(\psi) = \bar{\rho}(\psi)$ and for any $\phi \in \Phi, \phi \leq \psi \in \mathcal{E}$ implies $\rho(\psi) \leq \rho(\phi)$, hence $\bar{\rho}(\phi) \leq \rho(\phi)$. Then we have $\rho(\phi) = \bar{\rho}(\phi) \lor (\rho(\phi) - \bar{\rho}(\phi))$. It then follows that

$$sp_\nu(\phi) = \mu(\rho(\phi)) = \mu(\bar{\rho}(\phi)) + \mu((\rho(\phi) - \bar{\rho}(\phi)))$$

But from $sp_\nu = f = \mu(\bar{\rho}(\phi))$ we deduce that $\mu((\rho(\phi) - \bar{\rho}(\phi))) = 0$, hence $\rho(\phi) - \bar{\rho}(\phi) = \bot$. Since $\bar{\rho}(\phi) \leq \rho(\phi)$ this means that indeed $\bar{\rho}(\phi) = \rho(\phi)$. We may rephrase this result in the following Corollary.

**Corollary 10.2** If $\rho = \rho_0 \circ s_\nu$ is the allocation of probability associated with the support function $sp$ on $\mathcal{E}$ which has the least extension $sp_\nu = \mu \circ \rho$, then

$$\rho(\phi) = \bigvee \{ \rho(\psi) : \psi \in \mathcal{E}, \phi \leq \psi \}.$$  

If the support function $sp$ is defined on a lattice $\mathcal{E}$, then Theorem 10.15 may be sharpened (Shafer, 1973).
Theorem 10.16 If \( sp \) is a support function defined on a lattice \( E \subseteq \Phi \), then

\[
sp_{\nu}(\phi) = \sup\{sp(\psi) : \psi \in E, \phi \leq \psi\}. \tag{10.25}
\]

Proof. Since \( sp_{\nu} \) is monotone, the right hand side of (10.25) is less or equal to \( sp_{\nu} \). It remains to show the converse inequality. Again, let \( \rho = \rho_0 \circ sp_{\nu} \) be the a.o.p. associated with the support function \( sp \) and \( \mu \) the probability in the corresponding probability algebra \( (B, \mu) \). Consider \( \psi_1, \ldots, \psi_n \in E \). Since \( E \) is a lattice, \( \bigwedge_{i=1}^{n} \psi_i \) belongs to \( E \) too. Note that

\[
s_{\nu}(\bigwedge_{i=1}^{n} \psi_i) = \{ I \in I_\Phi : \bigwedge_{i=1}^{n} \psi_i \in \nu(I) \} \supseteq \bigcup_{i=1}^{n} \{ I \in I_\Phi : \psi_i \in \nu(I) \} = \bigcup_{i=1}^{n} s_{\nu}(\psi_i).
\]

Therefore,

\[
\rho(\bigwedge_{i=1}^{n} \psi_i) = [s_{\nu}(\bigwedge_{i=1}^{n} \psi_i)] \geq \bigcup_{i=1}^{n} s_{\nu}(\psi_i) = \bigvee_{i=1}^{n} \rho(\psi_i).
\]

Here \([A] \) denotes the projection of \( A \in A_\Phi \) to the associated Boolean algebra \( B \) in the probability algebra \( (B, \mu) \), see Section 9.4. Using (10.24) in the proof of Theorem 10.15 and \( sp_{\nu}(\phi) = f \), we obtain now

\[
sp_{\nu}(\Phi) = \sup\{\mu(\bigvee_{i=1}^{n} \rho(\psi_i)) \leq \sup\{\mu(\bigvee_{i=1}^{n} \rho(\psi_i))\}
\]

where the supremum ranges over \( \psi_i \in E, \phi \leq \psi_i, i = 1, \ldots, n \) and \( n = 1, 2, \ldots \). But, since \( E \) is a lattice, it follows that

\[
sp_{\nu} = \sup\{sp(\psi) : \psi \in E, \phi \leq \psi\} = \sup\{\mu(\rho(\psi)) : \psi \in E, \phi \leq \psi\}.
\]

This concludes the proof. \( \square \)

There are in particular several examples of compact information algebras where the finite elements form a lattice, hence where Theorem 10.16 applies if \( E = \Phi_f \). We give here a few examples

Example 10.1 Cofinite sets in a subset algebra evidently form a lattice, even a distributive one.

Example 10.2 Consider the information algebra of strings (see Example 3.1 in Section 3). Here we define meet between finite strings as follows:

\[
r \land s = \begin{cases} r, & \text{if } r \leq s, \\ s, & \text{if } s \leq r, \\ \epsilon, & \text{otherwise} \end{cases}
\]

This lattice is however not distributive (consider \( r \leq s \), and \( s, r \not\leq t \), then \( r \lor (s \land t) = r \), but \( (r \lor s) \land (r \lor t) = s \)). \( \square \)
Example 10.3 Convex polyhedra form a lattice, if meet is defined as the convex hull of the union of convex polyhedra. Again, this lattice is not distributive.

We have seen in Section 10.3, that support functions $sp$, defined on the finite elements $\Phi_f$ of a compact information algebra $(\Phi, D)$ may be extended either to a continuous support function $sp_\sigma$ or to a condensable one $sp_\gamma$. Further, by definition of condensability, $sp_\gamma$ is determined by the values of $sp$ on $\Phi_f$. This is like $sp_\sigma$, which according to Theorem ?? is also determined by the values of $sp$ on $\Phi_f$, if $E = \Phi_f$. Does a similar result also hold for the continuous extension $sp_\sigma$? Yes, but so far we know today, only for a very special case, namely if $E$ is a distributive lattice (Shafer, 1979), Theorem 4. The following theorem is a particular case of Shafer’s result, a case of special interest for us, where we assume that the finite elements form a distributive lattice, like the cofinite elements in a subset algebra.

Theorem 10.17 Let $(\Phi, D)$ be a compact information algebra, $(\Phi_f, D)$ a subalgebra of $(\Phi, D)$ and $\Phi_f$ a distributive lattice. If $sp$ is a support function defined on $\Phi_f$, then for all $\phi \in \Phi$,

$$sp_\sigma(\phi) = \sup \{ \lim_{i \to \infty} sp(\psi_i) : \psi_1 \leq \psi_2 \leq \ldots \in \Phi_f, \lor_{i=1}^{\infty} \psi_i \geq \phi \}. \quad (10.27)$$

Proof. We denote the right hand side of (10.27) by $f$. Note that $\lim_{i \to \infty} \psi_i = sp_\sigma(\lor_{i=1}^{\infty} \psi_i) \leq sp_\sigma(\phi)$ if $\lor_{i=1}^{\infty} \psi_i \geq \phi$. This shows that $sp_\sigma \geq f$. We are going to show that $f$ is a continuous support function extending $sp$. Since $sp_\sigma$ is the minimal continuous support function extending $sp$, this proves then that $sp_\sigma = f$.

Let $(\mu, B)$ be the probability algebra associated with the canonical probability space of the support function $sp$ and $\rho$ the corresponding allocation of probability, so that $= \mu \circ \rho$. For each $\phi \in \Phi$ define $D(\phi) \subseteq B$ by

$$D(\phi) = \{ \lambda_{i=1}^{\infty} \rho(\psi_i) : \psi_i \in \Phi_f, i = 1, 2, \ldots, \lor_{i=1}^{\infty} \psi_i \geq \phi \}$$

(here we follow again the proof of Theorem 4 in (Shafer, 1979)). The sets $D(\phi)$ are upwards directed: In fact, consider two countable sets $\psi_{1,i}, \psi_{2,j} \in \Phi_f$ such that $\lor_{i=1}^{\infty} \psi_{1,i}, \lor_{j=1}^{\infty} \psi_{2,j} \geq \phi$. Then, since $\Phi_f$ is a lattice, the set $\psi_{1,i} \land \psi_{2,j}$ is still countable in $\Phi_f$. And, since the lattice $\Phi_f$ is distributive, $\lor_{i,j=1}^{\infty} \psi_{1,i} \land \psi_{2,j} = (\lor_{i=1}^{\infty} \psi_{1,i}) \land (\lor_{j=1}^{\infty} \psi_{2,j}) \geq \phi$.

Finally, $\psi_{1,i} \land \psi_{2,j} \leq \psi_{1,i}, \psi_{2,j}$ implies $\rho(\psi_{1,i} \land \psi_{2,j}) \geq \rho(\psi_{1,i}), \rho(\psi_{2,j})$. So indeed, $D(\phi)$ is upwards directed.

Define now $\hat{\rho}(\phi) = \lor D(\phi)$. We claim that $\hat{\rho}$ is a $\sigma$-a.o.p. and that $f = \mu \circ \hat{\rho}$. This shows then that $f$ is a continuous support function. Since obviously $f|\Phi_f = sp$ this proves then the theorem.
It is evident that \( \tilde{\rho}(\emptyset) = \top \). So, it only remains to show that 
\( \tilde{\rho}(\bigvee_{i=1}^{\infty} \phi_i) = \wedge_{i=1}^{\infty} \phi_i \) or \( \bigvee D(\bigvee_{i=1}^{\infty} \phi_i) = \bigwedge_{i=1}^{\infty} \bigvee D(\phi_i) \). Fix a sequence \( \phi_1, \phi_2, \ldots \). To simplify notation let’s set \( D = D(\bigvee_{i=1}^{\infty} \phi_i) \) and \( D_i = D(\phi_i) \) and \( M = \bigwedge_i \vee D_i \). The task is then to show that 
\( \bigvee D = M \).

Now, \( D \subseteq D_i \) for all \( i \), hence \( \bigvee D \leq \bigwedge_i \vee D_i \). Further, since \( D_i \) are upwards directed sets, by Lemma 9.1 we have 
\[
\mu(\bigvee D_i) = \sup_{\psi \in D_i} \mu(\psi).
\]

Choose an \( \epsilon > 0 \). Then for all \( i \), there is an \( M_i \in D_i \) such that 
\[
\mu(\bigvee D_i - M_i) \leq \frac{\epsilon}{2^i}.
\]

Since \( M \leq \bigvee D_i \), we obtain also 
\[
\mu(M - M_i) = \mu(M \wedge M_i^c) \leq \mu(M_i \wedge M_i^c) = \mu(D_i - M_i) \leq \frac{\epsilon}{2^i}.
\]

Let \( B_i \) denote a set of elements \( \psi_1, \psi_2, \ldots \in \Phi_f \) such that \( \bigvee B_i \geq \phi_i \) and \( M_i = \wedge B_i \). Let \( B_{c\epsilon} = \bigcup_{i=1}^{\infty} B_i \) and \( M_{c\epsilon} = \wedge_{i=1}^{\infty} M_i \). Then \( \bigvee B_{c\epsilon} \geq \bigvee_{i=1}^{\infty} \phi_i \) and \( M_{c\epsilon} = \wedge_{i=1}^{\infty} B_i = \wedge_{\psi \in B_{c\epsilon}} \rho(\psi) \).

Thus \( M_{c\epsilon} \) belongs to \( D \), hence \( M_{c\epsilon} \leq \bigvee D \). We have 
\[
M - M_{c\epsilon} = M \wedge M_{c\epsilon} = M \wedge (\wedge_{i=1}^{\infty} M_i)^c = M \wedge (\bigvee_{i=1}^{\infty} M_i^c) = \bigvee_{i=1}^{\infty} (M \wedge M_i^c) = \bigvee_{i=1}^{\infty} (M - M_i).
\]

Thus we obtain 
\[
\mu(M - M_{c\epsilon}) = \mu(\bigvee_{i=1}^{\infty} (M - M_i)) \leq \epsilon.
\]

Now, \( M_{c\epsilon} \leq \bigvee D \) implies \( M_{c\epsilon}^c \geq (\bigvee D)^c \) and therefore \( M - \bigvee D = M \wedge (\bigvee D)^c \leq M \wedge M_{c\epsilon}^c = M - M_{c\epsilon} \). This shows that 
\[
\mu(M - D) \leq \mu(M - M_{c\epsilon}) \leq \epsilon.
\]

Since \( \epsilon \) is arbitrarily small, we conclude that \( \mu(M - \bigvee D) = 0 \) and from this it follows that \( \bigvee D = M \), because \( \bigvee D \leq M \). This proves that \( \tilde{\rho} \) is a \( \sigma \)-a.o.p.

Next, we are going to show that \( f = \mu \circ \tilde{\rho} \), hence that \( f \) is a continuous support function. Note that if \( sp_{\rho} = \mu \circ \rho \), then, since \( sp_{\sigma} | \Phi_f = sp \), we have 
\[
f(\phi) = \sup \{ \mu(\rho(\bigvee_{i=1}^{\infty} \psi_i)) : \psi_i \in \Phi_f, \bigvee_{i=1}^{\infty} \psi_i \geq (\phi) \} = \sup \{ \mu(\rho(\chi)) : \chi \in D(\phi) \}.
\]
This is so because $sp_\sigma$ is continuous, hence $\rho$ a $\sigma$-a.o.p. Since $D(\phi)$ is upwards directed, we obtain (Lemma 9.1)

$$f(\phi) = \mu(\vee D) = \mu(\tilde{\rho}(\phi))$$

This concludes the proof. 

Under the assumptions of Theorem 10.17 we may, according to the considerations in the proof, also write

$$sp_\sigma(\phi) = \sup\{sp_\sigma(\vee_{i=1}^\infty \psi_i) : \psi_i \in \phi_f, \vee_{i=1}^\infty \psi_i \geq \phi\},$$

or, equivalently,

$$sp_\sigma(\phi) = \sup\{sp_\sigma(\chi_i) : \chi_i \in \sigma(\phi_f), \chi \geq \phi\},$$

This shows that $sp_\sigma$ is fully determined by its values on finite elements. If $\Phi_f$ is in addition countable, then $\sigma(\Phi_f) = \Phi$ and

$$sp_\sigma(\phi) = \lim_{i \to \infty} sp_\phi$$

if $\psi_1 \leq \psi_2 \leq \ldots \in \phi_f$ and $\vee_{i=1}^\infty \psi_i = \phi$. According to (10.14), this result holds in general for countable $\Phi_f$, without additional assumptions.

We have now shown that a support function defined on some semilattice $E \subseteq \Phi$ of an information algebra $\Phi$ can have different kinds of extension, defined in terms of its values in $E$. Similar and more results of this kind can be found in (Shafer, 1979).

10.5 Plausibility Functions

10.6 The Boolean Case
References


Kohlas, J., & Wilson, N. 2006. *Exact and Approximate Local Computation in Semiring Induced Valuation Algebras*. Tech. rept. 06-06. Department of Informatics, University of Fribourg.


Scott, Dana. 1971. *Continuous Lattices*.


